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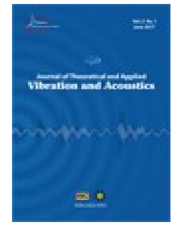

Exact analytical approach for free longitudinal vibration of nanorods based on nonlocal elasticity theory from wave standpoint

Mohammad Reza Hairi Yazdi *, **Mansour Nikkhah-Bahrami**, **Masih Loghmani**

School of Mechanical Engineering, College of Engineering, University of Tehran, Tehran, Iran

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ABSTRACT

In this paper, free longitudinal vibration of nanorods is investigated from the wave viewpoint. The Eringen's nonlocal elasticity theory is used for nanorods modelling. Wave propagation in a medium has a similar formulation as vibrations and thus, it can be used to describe the vibration behavior. Boundaries reflect the propagating waves after incident. Firstly, the governing equation of nanorods longitudinal vibration based on the Eringen's nonlocal elasticity theory is derived. Secondly, the propagation matrix for nanorod waveguide is derived and then the reflection matrix for spring boundary condition is calculated. The relations between amplitudes of propagation and reflection waves in the waveguide dominant are then combined in a matrix form format to set up a laconic efficient method for free axial vibration analysis of nanorods. The exact analytical solution for arbitrary boundary conditions natural frequencies is derived. To validate this approach, the exact solutions of special boundary conditions cases (clamped-clamped and clamped-free) are used. At the end, the effect of nonlocal parameter on the natural frequencies and boundary stiffness for arbitrary boundary condition is discussed.

1. Introduction

Innovation of carbon nanotubes [1] was a big inspiration for researchers in the field of nano-engineering and technology during recent years. The nonlocal elasticity theory was presented by Eringen [2-5]. According to this theory, the strain in every point of the structure affects the stress tensor for each particular point of the medium. The nonlocal elasticity theory has been addressed

* Corresponding author:
E-mail address: myazdi@ut.ac.ir (M.R. Hairi Yazdi)

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in several recent works. Longitudinal vibrations of the nanorods is studied by Aydogdu [6]. The nonlocal elasticity theory is applied on Euler–Bernoulli nanobeams to investigate the effects of nonlocal parameter on deflection by Peddieson et al. [7]. Aydogdu [8] studied the bending, buckling and vibration behaviors of nanobeams using a general nonlocal beam model. The effects of nonlocal parameter, boundary conditions and surrounded medium stiffness on natural frequencies of longitudinal vibration and buckling of embedded nanorods in an elastic medium are surveyed in several works [9-12]. Chang [13] investigated the longitudinal vibration of non-uniform and non-homogeneous nanorods. Aydogdu and Elishakoff [14] studied vibration of nanorods restrained by a linear spring. The effect of an attached buckyball as a concentrated mass at the tip of a single-walled-carbon-nanotube on longitudinal natural frequencies is investigated by Murmu and Adhikari [15]. Karličić et al. [16] considered the effects of a transversal magnetic field on free longitudinal vibration of a complicated multi-nanorod. Şimşek [17] solved the motion equation for free axial vibration of tapered nanorods in which density, Young modulus and cross-sectional area of the nanorod vary functionally along the nanorod length. The considered boundary conditions are clamped–clamped and clamped–free. Li et al. [18] showed there is a good agreement between atomistic simulations and nonlocal theoretical approaches for free vibration of nanorods. Narendar and Gopalakrishnan [19] developed a nonlocal rod model for coupled nanorods and have derived explicit expressions for wave number. They showed that considering the effect of small-length scale has a high impact on wave characteristics of nanorods which is ignored in the classic model of such rod. Huang [20] studied the effects of nonlocal parameter on axial vibration of nanorod when internal long-range interactions are included. He shows that the internal long-range interactions make the nanorod stiffer. Kiani [21] used the nonlocal theory to study the free axial vibration of tapered nanowires with varying cross-section as a polynomial function. He used a perturbation technique based on the Fredholm alternative theorem to assess the problem in a more general form. Vibration characteristics such as the natural frequencies and corresponding mode shapes are derived analytically. The results are in good agreement with the exact solution and the differences increase with the rate of radii change. Murmu and Adhikari [22] have studied the longitudinal vibration of a double-nanorod-system. The Eringen's nonlocal elasticity theory is utilized for the development of the governing equations. The two nanorods are coupled by longitudinally directed distributed springs. Clamped–clamped and clamped–free boundary conditions are considered and the explicit functions are derived. Li et al. [23] have derived an exact frequency equation for free axial vibration of a nanorod carrying a nanoparticle based on the nonlocal elasticity theory and the Love theory for longitudinal vibration. Oveissi et al. [24] investigated the effects of small-scale of the nanoflow and nanostructure on axial vibration of single-walled carbon fluid flowing nanotubes. They used the strain-inertia gradient and nonlocal theories for this study. According to this research, by increasing the Knudsen number and the effect of small-scale, the critical flow velocity will decrease. Using the strain-inertia gradient theory leads to such increase. Karličić et al. [25] used the Eringen's nonlocal theory for modelling a viscoelastic double-nanorod and investigated its free axial vibrations. They have found an exact solution for the longitudinal vibration of both clamped-clamped and clamped-free boundary conditions. The vibration of functionally graded nanorods has also been addressed by other researchers [26, 27].

Some research articles recently claimed that using the Eringen theory in the form of differential formulation may cause some inconsistencies. Challamel et al. [28] studied the self-adjointness of Eringen's nonlocal elasticity theory. A simple one-dimensional beam model is used for this purpose. They have observed the nonself-adjointess in the Eringen's model. Benvenuti and Simone [29] have shown that the integral approach of the Eringen theory has similar results in special cases as well as the Aifantis strain-gradient elasticity theory. Also, the small-scale effects in some cases can be captured by the Eringen theory when it is used in an integral form. The research is based on a one-dimensional elasticity problem.

Propagating waves in waveguides is an alternative expression for describing vibration in structures instead of mode combinations. Both of these approaches have their benefits for vibration analysis with different mathematical difficulties in different cases. Many researches deal with the waves' behaviors in structures such as propagation, transmission and reflection [30-34]. Mei [35] studied the control and vibration problem for axial oscillations in rods. Mei et al. [36] solved the motion equations for free and forced vibrations of axially pre-loaded cracked stepped Timoshenko beam analytically. Mei [37] studied the free lateral vibration of an Euler-Bernoulli beam with a mass at the tip. Mei and Sha [38] found an analytical solution for spatial structures which was in good agreement with experimental results. Mei [39] analyzed longitudinal vibrations of planar and L-shaped structures and designed an active controller for active discontinuities by using this method.

In this work, the governing motion equation for longitudinal vibration of nanorods according to the nonlocal elasticity theory will be extracted first. Then, we will apply the wave method to this equation to derive the propagation matrix so the reflection matrix for arbitrary boundary conditions will be calculated. Finally, an analytical explicit closed-form equation to obtain the axial natural frequencies will be expressed.

2. Nonlocal rod model for longitudinal vibration

As reported by the Eringen's nonlocal elasticity theory [2] the strain in all points of the structure affects the stress tensor of each particular point within the medium. The results of molecular dynamics validated the Eringen's assumption. According to this theory, the relation of stress–strain for homogeneous structures is:

$$\sigma_{ij}^{nl} = \int_V \varphi\left(\left|\mathbf{x}-\mathbf{x}'\right|,\eta\right)\sigma_{ij}^{l} d\,\mathbf{V} \tag{1}$$

where σ_{ij}^{nl} and σ_{ij}^{l} are the nonlocal and local stress tensors respectively. The integration is defined over the whole volume V. The small-scale effects is considered in the nonlocal modulus φ. The parameter φ depends on $\left|\mathbf{x}-\mathbf{x}'\right|$ and η where $\left|\mathbf{x}-\mathbf{x}'\right|$ is the distance between the points $\mathbf{x}$ and $\mathbf{x}'$, and $\eta = e_0 a / L$ with a and L as the internal and external lengths (e.g. the distance of two carbon molecules bond and the nanorod length). e_0 is a constant coefficient which corresponds to the material. The function φ is defined as:

$$\varphi\left(\left|\mathbf{x}\right|,\eta\right) = \left(2\pi L^2\eta^2\right)^{-1} K_0\left(\frac{\sqrt{\mathbf{x}.\mathbf{x}}}{L\eta}\right) \tag{2}$$

in which K_0 is the modified Bessel function. Because of the difficulty in applying Eq. (1), the following equivalent equation can be used [7]:

$$\left(1-(e_0 a)^2 \nabla^2\right)\sigma_{ij}^{nl} = C:\varepsilon \tag{3}$$

where C and ε are the elasticity and strain tensors respectively. The double dot product is denoted by ":" in this equation . ∇^2 is the Laplacian operator. For one-dimensional problems, the constitutive relation is obtained as:

$$\left(1-(e_0 a)^2 \frac{\partial^2}{\partial x^2}\right)\sigma_{ij}^{nl} = E\varepsilon_{xx} \tag{4}$$

where E is the Young modulus. The governing equation of motion for longitudinal vibration of nanorods is:

$$\frac{\partial N^l}{\partial x} = mA(x)\frac{\partial^2 u(x,t)}{\partial t^2} \tag{5}$$

where $u(x,t)$ and m are the longitudinal displacement and mass density respectively. N^l is the axial force and is obtained as:

$$N^l = \int_A \sigma_{xx}^l dA \tag{6}$$

where A is the area of cross section. Combining Eqs. (4) and (6) leads to:

$$N^{nl} - (e_0 a)^2 \frac{\partial^2 N^{nl}}{\partial x^2} = N^l \tag{7}$$

By using Eqs. (4-7), equation of motion for free longitudinal vibration of nanorods will be obtained as:

$$(e_0 a)^2 m \frac{\partial^4 u(x,t)}{\partial x^2 \partial t^2} + E\frac{\partial^2 u(x,t)}{\partial x^2} - m\frac{\partial^2 u(x,t)}{\partial t^2} = 0 \tag{8}$$

When $e_0 a = 0$, Eq. (8) reduces to the equation of motion for the classical rod.

3. Wave approach

3.1. Wave description and analysis of longitudinal vibrations in a classic rod [27]

The equation of motion for a longitudinally vibrating uniform classic rod is described by:

$$m\frac{\partial^2 u(x,t)}{\partial t^2} - E\frac{\partial^2 u(x,t)}{\partial x^2} = 0 \tag{9}$$

In the absence of the external forces, the homogeneous Eq. (9) has a solution that can be written as the sum of two longitudinal wave components as:

$$u_{cl}(x) = \left(a^+ e^{-ik_{cl}x} + a^- e^{ik_{cl}x}\right)e^{i\omega t} \tag{10}$$

where a^+ and a^- are the positive and negative travelling wave amplitudes respectively, k_{cl} is the wavenumber and index cl denotes the classic rod model. The wave number can be calculated by substituting Eq. (10) into Eq. (9) as:

$$k_{cl} = \sqrt{m/E}\,\omega \tag{11}$$

In the absence of discontinuities, the wave amplitudes along a uniform length of rod are related by (Fig. 1):

$$\left.\begin{aligned} a^+(x_0+x) &= F^+(x)a^+(x_0) \\ a^-(x_0) &= F^-(x)a^-(x_0+x) \end{aligned}\right| \Rightarrow F^+(x) = F^-(x) = F(x) = e^{-ik_{cl}x} \tag{12}$$

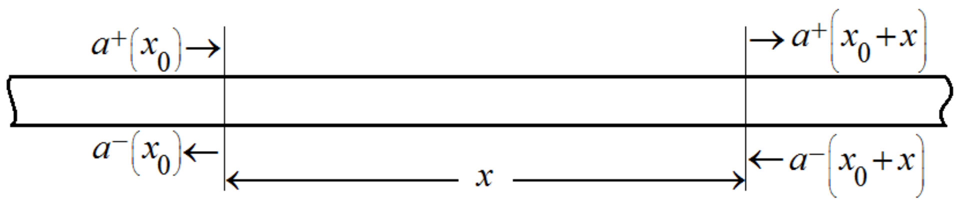

Fig. 1. Amplitudes of wave in two points of rod with distance x

where $F^+(x) = F^-(x) = F(x) = e^{-ik_{cl}x}$ is the propagation matrix. The axial force $P_{cl}(x,t)$ is:

$$P_{cl}(x,t) = EA\frac{\partial u_{cl}(x,t)}{\partial x} = EA\left[-ikC_1e^{-ik_{cl}x} + ikC_2e^{ik_{cl}x}\right]e^{i\omega t} \tag{13}$$

Fig. 2 shows a common general boundary. The relation between incident a^+ and reflected a^- waves is:

$$a^- = ra^+ \tag{14}$$

For each general boundary (Fig. 2) the reflection matrix r can be obtained by applying the equilibrium equation at the boundary:

$$P = -\bar{K}_T u \tag{15}$$

Substituting Eqs. (10) and (13) in Eq. (15) results into,

$$a^+ - a^- = -iK_T\left(a^+ + a^-\right) \tag{16}$$

in which $K_T = \bar{K}_T / EAk_{cl}$ is the dimensionless stiffness.

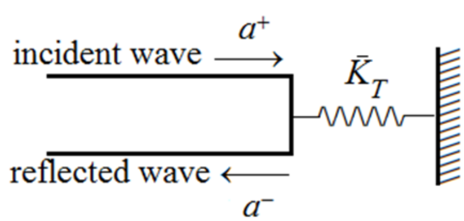

Fig. 2. General boundary

Rearranging Eq. (16) and considering Eq. (14) yields to:

$$r = \left[\frac{1 + iK_T}{1 - iK_T} \right] \tag{17}$$

When the boundary is clamped, K_T becomes very large ($K_T \to \infty$) and thus $r = [-1]$. For free boundary condition, $K_T \to 0$ and then $r = [1]$

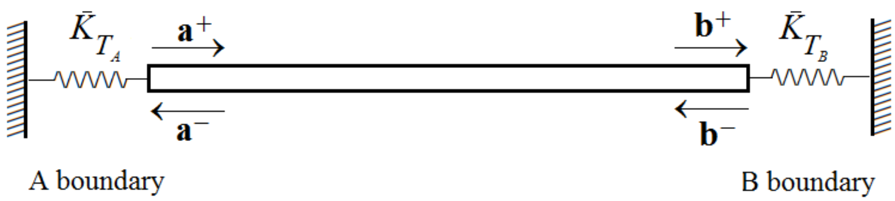

Fig. 3. (Nano-Classic) rod with spring boundary conditions

Fig. 3 shows a classic rod. a^- and b^+ are the incident waves at the boundaries A and B respectively and a^+ and b^- are the corresponding reflected waves. The incident and corresponding reflected waves at the boundaries A and B are denoted by $a^{\mp}$ and $b^{\pm}$ respectively. According to Eqs. (12) and (14), we have $b^+ = F(L)a^+$ and $a^- = F(L)b^-$ for propagation relations and $a^+ = r_A a^-$ and $b^- = r_B b^+$ for reflection relations. Rewriting these four equations in the matrix form leads to:

$$\begin{bmatrix} -1 & r_A & 0 & 0 \\ F(L) & 0 & -1 & 0 \\ 0 & -1 & 0 & F(L) \\ 0 & 0 & r_B & -1 \end{bmatrix} \begin{Bmatrix} a^+ \\ a^- \\ b^+ \\ b^- \end{Bmatrix} = 0 \tag{18}$$

To have a nontrivial solution, the determinant of Eq. (18) should be equal to zero so the eigenvalue equation will be obtained analytically as:

$$F(L)^2 r_A r_B - 1 = 0 \tag{19}$$

By substituting $F(L)$ from Eq. (12) and setting $r_A = -1$ and $r_B = 1$ for clamped- free boundary conditions, Eq. (19) reduces to:

$$e^{-2ik_{cl}L} + 1 = 0 \tag{20}$$

The eigenvalue equation then becomes:

$$k_{cl_n} L = \frac{(2n-1)}{2}\pi, \qquad n = 1, 2, \ldots \tag{21}$$

By defining the dimensionless frequency as:

$$\Omega = \omega L\sqrt{m / E} \tag{22}$$

Substituting Eq. (11) into Eq. (21) yields to:

$$\Omega_{cl_n} = \frac{(2n-1)}{2}\pi, \qquad n = 1,2,... \tag{23}$$

and for clamped-clamped boundary conditions, by substituting $F(L)$ from Eq. (12) and setting $r_A = -1,\ r_B = -1$ for clamped- clamped boundary condition Eq. (19) reduces to:

$$e^{-2ik_{cl}L} - 1 = 0 \tag{24}$$

The eigenvalue equation will then be:

$$k_{cl_n} L = n\pi, \qquad n = 0,1,2,... \tag{25}$$

Finally by substituting the wave number k_{cl} form Eq. (11) into Eq. (25), the natural frequencies simplify to:

$$\Omega_{cl_n} = n\pi, \qquad n = 1,2,... \tag{26}$$

3.2. Wave description of nanorod

To solve free longitudinal vibration Eq. (8) for any given arbitrary boundary conditions we use the method of separation of variables which is:

$$u_{na}(x,t) = U(x)G(t) \tag{27}$$

$U(x)$ and $G(t)$ are the displacement and time terms respectively and Eq. (26) can be written as:

$$u_{na}(x,t) = \left[D_1 e^{-ik_{na}x} + D_2 e^{ik_{na}x}\right]e^{i\omega t} = \left(D_1 e^{-i(k_n ax-\omega t)} + D_2 e^{i(k_{na}x+\omega t)}\right) \tag{28}$$

where k_n and ω are the wave number and natural frequency respectively and the index na denotes the nano model. The above equation can be written as:

$$u_{na}(x,t) = \left(a^+ + a^-\right)e^{i\omega t}, \quad a^+ = D_1 e^{-ik_{na}x}, \quad a^- = D_2 e^{ik_{na}x} \tag{29}$$

in which $a^+ = D_1 e^{-ik_{na}x}$ and $a^- = D_2 e^{ik_{na}x}$ represent positive-going and negative-going amplitudes by velocity (ω / k_{na}) respectively. By substituting Eq. (28) into Eq. (8) we have:

$$k_{na} = \sqrt{\frac{m\omega^2}{E - m(e_0 a)^2 \omega^2}} \tag{30}$$

The axial force $P_{na}(x,t)$ is:

$$P_{na}(x,t) = EA\frac{\partial u_{na}(x,t)}{\partial x} = EA\left[-ikC_1 e^{-ik_{na}x} + ikC_2 e^{ik_{na}x}\right]e^{i\omega t} \tag{31}$$

Eqs. (28) and (31) can be written in the matrix form as:

$$\begin{Bmatrix} U(x) \\ P(x) \end{Bmatrix} = \begin{bmatrix} \psi^+ & \psi^- \\ \varphi^+ & \varphi^- \end{bmatrix} \begin{Bmatrix} a^+ \\ a^- \end{Bmatrix} \tag{32}$$

in which

$$\begin{aligned} \psi^+ &= [1] & \psi^- &= [1] \\ \varphi^+ &= [-ikEA] & \varphi^- &= [ikEA] \end{aligned} \tag{33}$$

The propagation matrices F is as defined in Eq. (12) respects to its own wave number k_{na} which is defined in Eq. (30). By considering Eqs. (33) and (32), the Eq. (15) becomes:

$$\varphi^+ a^+ + \varphi^- a^- = -\bar{K}_T \left(\psi^+ a^+ + \psi^- a^- \right) \tag{34}$$

Therefore the reflection matrix Eq. (14) will become:

$$r = \left[-\frac{\varphi^+ + \bar{K}_T \psi^+}{\bar{K}_T + \psi^- \varphi^-} \right] \tag{35}$$

Eq. (35) can be simplified to Eq. (17) that is determined for classic rods.

4. Vibration analysis

In this section, the natural frequency equations for the two cases of arbitrary boundary conditions will be derived by combining the propagation and corresponding reflection matrices in each case. This equation has closed-form solution for any arbitrary boundary condition and special cases such as clamped-clamped and clamped-free boundaries.

4.1. General solution

Eq. (19) is the general solution form which is derived in the previous section for a classic rod model. It is also applicable for nanorods. To use the above equation, it is necessary to consider the appropriate nano-functions of wave number k_{na} and reflection matrix which are obtained in Eqs. (33) and (35) respectively. Without loss of generality for spring-spring boundary conditions, this equation will be simplified for the cases of clamped-spring and free-spring boundary conditions. At the next subsection, the two special cases of clamped-free and clamped-clamped will be considered.

Once the left side is clamped, it means $r_A = -1$ and the right side has an arbitrary boundary condition. Eq. (19) will be rewritten as:

$$F(L)^2 r_B + 1 = 0 \tag{36}$$

By substituting $F(L)$ from Eq. (12) and r_B from Eq. (17), Eq. (36) becomes:

$$e^{-2ik_{na}L} = -\frac{1 - iK_T}{1 + iK_T} \tag{37}$$

Therefore, the solution for the clamped-arbitrary boundary is:

$$k_{na}L = \frac{1}{2}\cos^{-1}\left[-\frac{1-K_T^2}{1+K_T^2}\right] \tag{38}$$

Finally, the dimensionless frequencies for the clamped-arbitrary boundary condition will be driven as:

$$\Omega_{na} = \frac{\cos^{-1}\left[-\frac{1-K_T^2}{1+K_T^2}\right]}{\sqrt{4+\left(\frac{e_0 a}{L}\right)^2\left(\cos^{-1}\left[-\frac{1-K_T^2}{1+K_T^2}\right]\right)^2}} \tag{39}$$

The first dimensionless natural frequency versus the dimensionless stiffness K_T for different nonlocal parameters e_0a/L are shown in Fig. 4. Also, Fig. 5 shows the first three natural frequencies for the nonlocal parameter $e_0a/L = 0.5$ versus the dimensionless stiffness K_T.

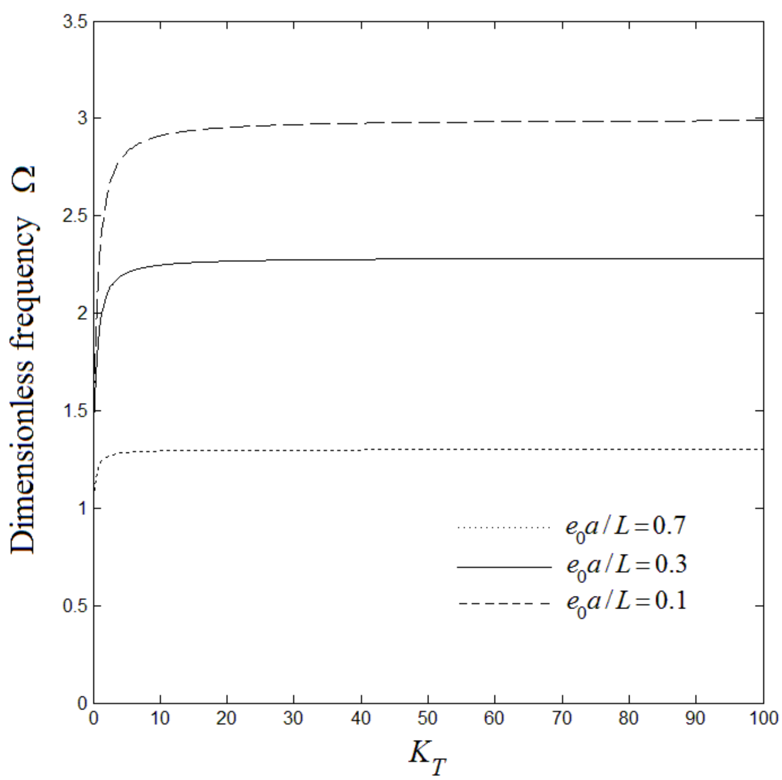

Fig. 4. The first dimensionless frequency vs. the dimensionless stiffness K_T for different nonlocal parameters e_0a/L in clamped-arbitrary boundary conditions

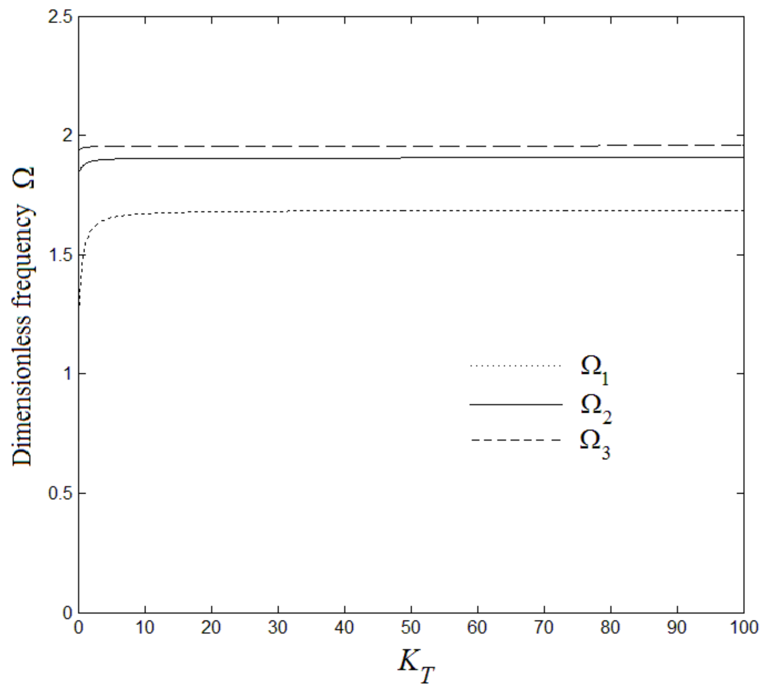

Fig. 5. The first three natural frequencies for the nonlocal parameter $e_0a/L=0.5$ vs. the dimensionless stiffness K_T in clamped-arbitrary boundary conditions

Once the right side is free, it means $r_A=1$ and the right side has an arbitrary boundary condition and Eq. (19) will be reframed as:

$$F(L)^2 r_B - 1 = 0 \tag{40}$$

By substituting $F(L)$ from Eq. (12) and r_B from Eq. (17), Eq. (40) becomes:

$$e^{-2ik_{na}L} = \frac{1-iK_T}{1+iK_T} \tag{41}$$

So the solution for free-arbitrary boundary conditions is:

$$k_{na}L = \frac{1}{2}\cos^{-1}\left[\frac{1-K_T^2}{1+K_T^2}\right] \tag{42}$$

Finally, the dimensionless frequencies of Eq. (41) for free-arbitrary boundary condition will be driven as:

$$\Omega_{na} = \frac{\cos^{-1}\left[\frac{1-K_T^2}{1+K_T^2}\right]}{\sqrt{4+\left(\frac{e_0a}{L}\right)^2\left(\cos^{-1}\left[\frac{1-K_T^2}{1+K_T^2}\right]\right)^2}} \tag{43}$$

The variations of the first dimensionless natural frequency versus the dimensionless stiffness K_T for different nonlocal parameters e_0a/L are shown in Fig. 6. Fig. 7 shows the first three natural frequencies for the nonlocal parameter $e_0a/L = 0.5$ versus the dimensionless stiffness K_T.

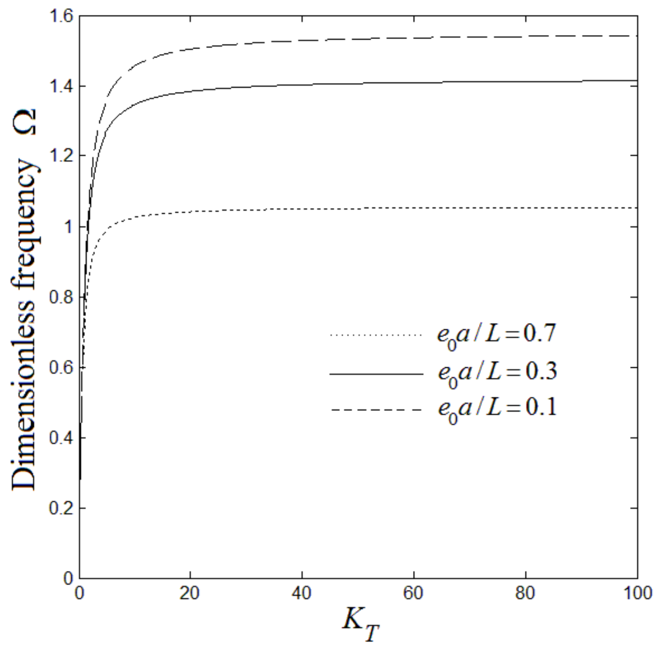

Fig. 6. The first dimensionless frequency vs. dimensionless stiffness K_T for different nonlocal parameters e_0a/L in free-arbitrary boundary conditions

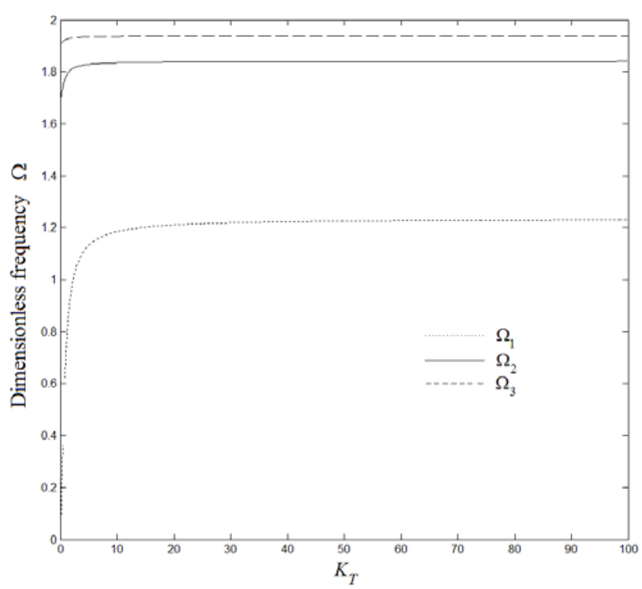

Fig. 7. The first three natural frequencies for nonlocal parameter $e_0a/L = 0.5$ vs. the dimensionless stiffness K_T in free-arbitrary boundary conditions

4.2. Special cases

In this section, clamped-clamped and clamped-free boundary conditions are considered as special cases. For clamped-clamped boundaries, the reflection matrices at A and B became $r_A = r_B = -1$ so Eq. (19) simplifies to:

$$e^{-2ik_{na}L} - 1 = 0 \tag{44}$$

An explicit exact closed-form equation for the clamped-clamped case will be obtained as,

$$k_{na}L = n\pi, \qquad n = 1,2,... \tag{45}$$

By substituting k_{na} from Eq. (30) the dimensionless frequencies for clamped-clamped boundary becomes:

$$\Omega_{na_n} = \frac{n\pi}{\sqrt{1+\left(e_0a/L\right)^2(n\pi)^2}}, \qquad n = 1,2,... \tag{46}$$

In Fig. 8(a) the first three dimensionless frequencies versus the nonlocal parameter e_0a/L for clamped-clamped boundary conditions are shown.

For clamped-free boundaries, the reflection matrices at A and B become $r_A = -1$ and $r_B = 1$. Therefore, Eq. (19) simplifies to,

$$e^{-2ik_{na}L} + 1 = 0 \tag{47}$$

The closed-form eigenvalue equation for clamped-free boundary condition is then obtained:

$$k_{na}L = \frac{(2n-1)}{2}\pi, \qquad n = 1,2,... \tag{48}$$

which yields to,

$$\Omega_{na_n} = \frac{(2n-1)(\pi/2)}{\sqrt{1+\left(e_0a/L\right)^2\left[(2n-1)(\pi/2)\right]^2}}, \quad n = 1,2,... \tag{49}$$

The first three dimensionless frequencies versus the nonlocal parameter e_0a/L are shown in Fig. 8(b).

In Table 1 the dimensionless natural frequencies for different nonlocal parameters are calculated from the wave approach and they are compared with the results from Aydogdu [6] for clamped-clamped boundary conditions.

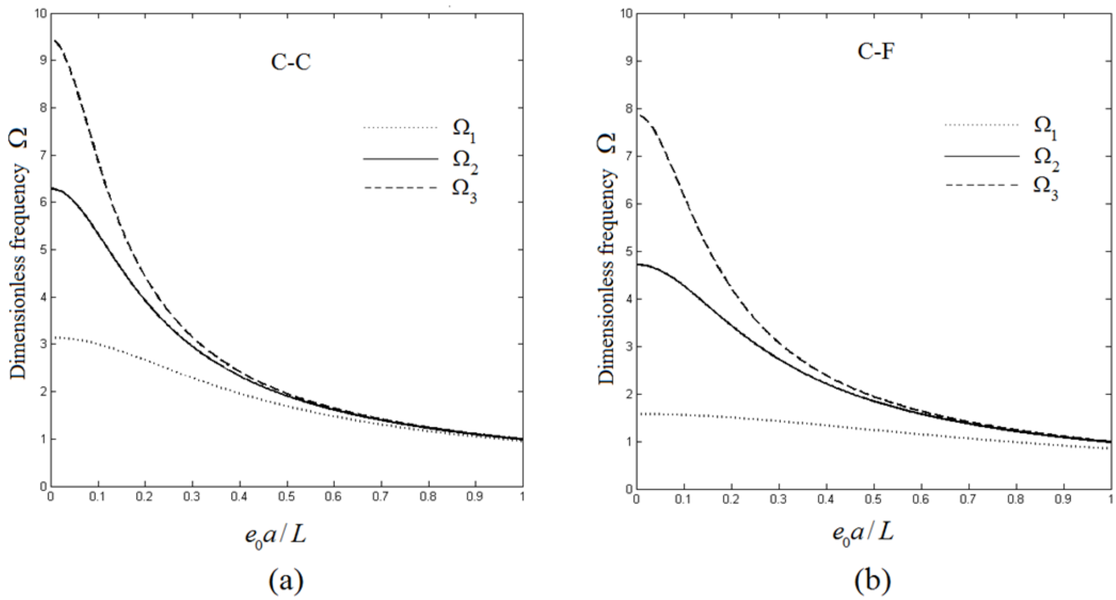

Fig. 8. The first three dimensionless frequencies vs. the nonlocal parameter e_0a/L for (a) C-C boundary condition (b) C-F boundary condition

Table 1. The dimensionless natural frequencies for different nonlocal parameters, a comparison between Aydogdu [6] and the wave approach for C-C boundary conditions

	$e_0a/L=0.1$		$e_0a/L=0.3$		$e_0a/L=0.5$		$e_0a/L=0.7$	
Ω	**Aydogdu**	**Wave approach**	**Aydogdu**	**Wave approach**	**Aydogdu**	**Wave approach**	**Aydogdu**	**Wave approach**
1	2.9972	2.9972	2.2862	2.2862	1.6871	1.6871	1.3004	1.3004
2	5.3201	5.3201	2.9446	2.9446	1.9058	1.9058	1.3930	1.3930
3	6.8587	6.8587	3.1426	3.1426	1.9564	1.9564	1.4124	1.4124
4	7.8248	7.8248	3.2219	3.2219	1.9751	1.9751	1.4194	1.4194
5	8.4356	8.4356	3.2607	3.2607	1.9840	1.9840	1.4227	1.4227
10	9.5289	9.5289	3.3147	3.3147	1.9960	1.9960	1.4271	1.4271
100	9.9949	9.9949	3.3331	3.3331	2	2	1.4286	1.4286

In Table 2 the dimensionless natural frequencies for different nonlocal parameters are calculated from the wave approach and compared with the results from Aydogdu [6] for clamped-free boundary conditions.

Table 2. The dimensionless natural frequencies for different nonlocal parameters, a comparison between Aydogdu [6] and the wave approach for C-F boundary conditions.

	$e_0a/L=0.1$		$e_0a/L=0.3$		$e_0a/L=0.5$		$e_0a/L=0.7$	
Ω	**Aydogdu**	**Wave approach**	**Aydogdu**	**Wave approach**	**Aydogdu**	**Wave approach**	**Aydogdu**	**Wave approach**
1	1.5518	1.5518	1.4209	1.4209	1.2353	1.2353	1.0569	1.0569
2	4.2628	4.2628	2.7213	2.7213	1.8411	1.8411	1.3671	1.3671
3	6.1767	6.1767	3.0684	3.0684	1.9381	1.9381	1.4055	1.4055
4	7.3981	7.3981	3.1900	3.1900	1.9677	1.9677	1.4167	1.4167
5	8.1440	8.1440	3.2444	3.2444	1.9803	1.9803	1.4213	1.4213
10	9.4819	9.4819	3.3127	3.3127	1.9956	1.9956	1.4269	1.4269
100	9.9949	9.9949	3.3331	3.3331	2	2	1.4286	1.4286

Tables (1) and (2) show the same results for Aydogdu [6] solution and the wave approach and also Eqs. (47) and (50) are in full agreement with the existing method [6].

5. Discussion and conclusion

Figures (4-7) show that by increasing the dimensionless stiffness K_T for different nonlocal parameters, natural frequencies increase and go from clamped-free to clamped-clamped and from free-free to clamped-free boundary conditions. It can be concluded from Fig. 5 and Fig. 7 that increasing the boundary stiffness has more effect on the lower natural frequencies. The increase in natural frequencies due to increasing the arbitrary boundary stiffness, for lower stiffness is more than for other amounts of the boundary stiffness. By decreasing the nonlocal parameter e_0a/L, the natural frequencies become closer to each other.

As shown in Eqs. (46) and (49) and also Fig. (8), by increasing the nonlocal parameter e_0a/L, the natural frequency will decrease and the frequencies have the same amounts as the classic model in Eqs. (23) and (26) when this term vanishes. Increasing the nonlocal parameter causes more decrease in the natural frequencies for higher frequencies. In addition, natural frequencies become closer to each other by increasing the nonlocal parameter. According to Fig. 8, in all natural frequencies there is an inflection point that in higher modes occurs in lower nonlocal parameters. This means decreasing the natural frequencies by increasing the nonlocal parameter has more speed after passing a critical amount of nonlocal parameter. Tables (1) and (2) show that the natural frequencies become closer for clamped-clamped and clamped-free cases in higher modes and in each mode they are independent from the nonlocal parameter.

It has been shown that by considering vibrations as moving waves in structures, a new viewpoint for vibrational analysis is opened. Combining the derived matrices for propagation and reflection, leads us to a pithy analytical approach to study longitudinal vibration of nanorods. This method is easily applicable for any arbitrary boundary condition. Simpler mathematical operations in such approach is one of its noticeable benefits as compared with the other existing methods.

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