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An analytical approach for the nonlinear forced vibration of clamped-clamped buckled beam

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ABSTRACT

Analytical solutions are attractive for parametric studies and consideration of the problems physics. In addition, analytical solutions can be employed as a reference framework for verification of numerical results. In this paper Homotopy analysis method and Homotopy Pade technique which are approximate analytical methods, are used to obtain nonlinear forced vibration response of Euler-Bernoulli clamped-clamped buckled beam subjected to an axial force and transverse harmonic load for the first time. Analytical solutions for nonlinear frequency are derived via Homotopy analysis method, Homotopy Pade technique and Runge Kutta method and the results are compared with experimental results of literature. Also the time response of the beam is obtained for free and forced vibration via analytical and numerical methods. In addition, the frequency response is drawn. Comparison of analytical results with numerical results and literature results reveals that Homotopy analysis method and Homotopy Pade technique have excellent accuracy for wide range of nonlinear parameters and predict system behavior precisely.

1. Introduction

Beams construct variety of micro/nano and macro dimensions structures such as micro oscillators, micro/nano resonators, airplane wings, flexible satellites, helicopter rotor blades, spacecraft antennae and long span bridges. Increase of oscillation amplitude leads to nonlinear behavior which cause fatigue phenomenon and structural breakdown. Increase of oscillation

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amplitude is large around the natural frequencies of structures [1]. Therefore, an accurate nonlinear vibration study of structures is substantial.

The nonlinear vibration of beams is formulated by nonlinear partial-differential equation in time and space with various boundary conditions. Researchers have been focused on approximate analytical techniques[2-10] and numerical methods [11-13] to solve nonlinear equations.

Although numerical methods such as finite element and boundary element methods have some advantages, analytical solutions appear more attractive for parametric studies and considering the problems physics. Also, analytical solutions are employed as a reference framework for verification and validation of numerical results.

On the whole, analytic methods have some restrictions. For example, perturbation methods are limited to weak nonlinear problems and related to a small parameter variation in the equation. Most of nonlinear problems, especially those with strong nonlinearity, don't have a small parameter variation. In order to prevail this problem, Homotopy analysis method (HAM) was proposed which requires no small parameter variation in the equation at all [14-17]. Also, Homotopy Pade method which is a combination of Pade approximation, the best approximation of fractional functions, and HAM was introduced [18, 19]. The effectiveness of HAM and Homotopy Pade method have been investigated in the analysis of different nonlinear problems. Kargarnovin *et al.* [20] have used HAM to get analytical solution for nonlinear free vibration of Euler-Bernoulli, Rayleigh, Shear and Timoshenko beams with pinned-pinned ends. Solutions for natural frequencies, beam deflection, critical buckling load and post-buckling load-deflection relation have been obtained. The verification of results by literature demonstrated good agreement between them. Pirbodaghi *et al.* [21] have employed HAM to study nonlinear free vibration analysis of Euler-Bernoulli beams subjected to axial loads. The effect of vibration amplitude on the buckling load and nonlinear frequency has been presented. The results follow available results in the literature precisely. Pirbodaghi *et al.* [22] have used HAM and Homotopy Pade technigue for solving Duffing equation with cubic and quantic nonlinearities. Comparison of obtained results with numerical results demonstrated very good agreement. Fooladi *et al.* [23] have applied HAM to solve the problem of Kirchhoff simplified model for beam. Comparison of the obtained results with the numerical solutions such as shooting method and fourth order Runge Kutta method have revealed that this method is efficient for the solution of this problem. Hoseini *et al.* [24] have employed HAM and Homotopy Pade technique to investigate analytic solution of fundamental nonlinear natural frequency and its corresponding displacement for tapered beams. The results have been verified against numerical solutions. Motallebi *et al.*[25] have used HAM and Homotopy Pade technique to analyze free nonlinear vibration of a simply supported Euler-Bernoulli beam under axial force. The effects of axial force and slenderness ratio on natural frequency and also the effects of nonlinear factors on the time response of beam have been investigated. The results have been verified using forth order Runge-Kutta method results. Jafari-Talookaei *et al.*[26] have investigated the nonlinear free vibration of the Euler-Bernoulli, Rayleigh, Shear and Timoshenko beams with simply supported boundary conditions using HAM. Expressions have been presented for the natural frequencies, the transverse deflection, critical buckling load and postbuckling load-deflection. An excellent agreement has been shown among the results of nonlinear analysis and the published results in literature. Fereidoon *et al.* [27] have investigated nonlinear vibration response of buckled beam subjected to axial load via homotopy perturbation method. Lacarbonara *et al.*[28] have analyzed nonlinear planar vibrations of a clamped-clamped buckled beam about its first post-buckling shape by multiple scales method. Also frequency-response curves are contrasted with experimental results. Abou-Rayan *et al.* [29]. have done nonlinear analysis of a simply supported buckled beam subjected to a harmonic axial load via multiple scales method. Afandeh *et al*. [30] have obtained nonlinear solution of an initially buckled beam via analytical, numerical and experimental methods. The method of multiple time scales has been applied to derive the equations. Ramu *et al*. [31] have studied the bifurcation behavior of a pre-buckled beam. Kreider *et al*. [32] have studied single-mode responses of a fixed-fixed buckled beam which is under a uniform, harmonic, transverse excitation. Nayfeh *et al*. [33] have obtained linear modes of buckled beams vibration through analytical and experimental methods. Ji *et al*. [34] have studied the nonlinear response of a clamped-sliding post-buckled beam subjected to a harmonic axial load experimentally. Lestari *et al*. [35] have obtained precise solution of the nonlinear dynamics for buckled beams with various boundary conditions. Min *et al*. [36] have studied the stability and steady state response of free and forced vibration of axially restrained, simply supported buckled beam. Tseng *et al*. [37] have studied a fixed-fixed buckled beam excited by the harmonic motion of its supporting base by analytical and experimental methods. Lacarbonara *et al*. [38] has investigated nonlinear planar vibration of a buckled beam around its first buckling mode configuration. He has compared obtained results with experimental data. Eisley [39] has studied vibration about the static buckled position for buckled beam or plate for a case of forced motion. Smelova-Reynolds *et al*. [40] have combined chaotic motion study with analytic technique in nonlinear dynamics for buckled beam. The have calculated the first component of the Melnikov vector. Smelova-Reynolds *et al*. [41] have found components of the Melnikov vector to demonstrate the change of critical condition for the whole Melnikov vector. Tang *et al*. [42] have investigated chaotic oscillations of a buckled beam subjected to forced external excitation. The results of numerical simulations have been compared with experimental data.

In this paper, for the first time HAM and Homotopy Pade method are used to analyze forced nonlinear vibration of a clamped-clamped buckled beam subjected to axial force and transverse harmonic load. To do this, first partial differential equation of beam is reduced to a typical nonlinear differential equation via Galerkin decomposition technique. The responses of HAM and Homotopy Pade method are compared with that of the Runge-Kutta numerical method and also related results of the literature. Also, the frequency response of HAM is drawn.

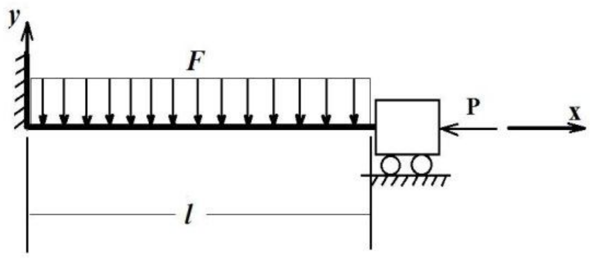


Fig.1. A schematic of clamped-clamped beam under transverse load and axial force.

2. Problem formulation

Fig. 1 shows a clamped-clamped beam of length l, cross sectional area A, moment of inertia I and young modulus E subjected to a constant axial force P and distributed transverse load $\mathrm{F} = f \cos kt$ where f and k are respectively amplitude and frequency of the distributed load. Euler Bernoulli beam theory has been used to obtain differential equation of motion. The beam cross-sectional area is considered to be uniform and its material is homogeneous.

The partial differential equation of motion for the transverse vibration of the beam is as follows [35],

$$\mu \frac{\partial^2 \mathrm{w}}{\partial t^2} + EI \frac{\partial^4 \mathrm{w}}{\partial x^4} + P \frac{\partial^2 \mathrm{w}}{\partial x^2} - \frac{EA}{2l} \frac{\partial^2 \mathrm{w}}{\partial x^2} \int_0^l \left(\frac{\partial \mathrm{w}}{\partial x} \right)^2 \mathrm{d}x = f \cos kt \tag{1}$$

where μ, w, x and t are respectively the mass per unit length of the uniformed beam, the transverse displacement, the longitudinal coordinate and the time. The first mode shape of the beam is [35]:

$$\phi(\mathrm{x}) = \cos(\frac{\alpha \mathrm{x}}{l}) - \cosh(\frac{\alpha \mathrm{x}}{l}) + \frac{\sin(\alpha) + \sinh(\alpha)}{\cos(\alpha) - \cosh(\alpha)} (\sin(\frac{\alpha \mathrm{x}}{l}) - \sinh(\frac{\alpha \mathrm{x}}{l})) \tag{2}$$

For a beam with clamped-clamped boundary conditions α is considered to be 4.730. Assuming $\mathrm{w}(x,t) = \hat{a} q(t) \phi(x)$ where $\mathrm{q}(t)$ and $\hat{a}$ are respectively time response of the beam and arbitrary constant for the amplitude of deflection and then applying the Galerkin method leads to the equation of motion of. [35]:

$$\ddot{\mathrm{q}} + \omega^2 \mathrm{q} + \gamma \mathrm{q}^3 = \tilde{f} \cos kt \tag{3}$$

$$\mathrm{q}(0) = a_0 = \frac{w_{\max}}{\hat{a}\phi(0.5l)}, \quad \frac{\mathrm{dq}}{\mathrm{d}t}(0) = 0 \tag{4}$$

where $a_0$is initial displacement which is dependent to the beam maximum deflection (w_{max}). Also ω and γ are respectively linear frequency and beam nonlinear parameter which are obtained in the following form[35]:

$$\omega^2 = \frac{EI}{m} \frac{\int_0^l \phi^{(iv)} \phi dx}{\int_0^l \phi^2 dx} + \frac{P}{m} \frac{\int_0^l \phi'' \phi dx}{\int_0^l \phi^2 dx} \tag{5}$$

$$\gamma = -\frac{EI\hat{a}^2}{2mlr^2} \frac{\int_0^l \phi'' \phi dx \int_0^l (\phi')^2 dx}{\int_0^l \phi^2 dx} \tag{6}$$

$$\tilde{f} = \int_0^l \frac{f}{m} \phi \mathrm{d}x \tag{7}$$

where $r = \sqrt{I/A}$ demonstrates the gyration radius for the beam cross-section.

3. Homotopy analysis method

One of the precise approximate analytical methods for nonlinear differential equations solution is Homotopy Analysis Method (HAM). The HAM embeds an auxiliary parameter p to transform a nonlinear differential equation into unlimited number of linear differential equations. Typical range of p is from zero to one. As p increases from zero to one, solution of the problem moves from the initial guess to the precise answer. The Homotopy function is defined as [15]:

$$\overline{\mathrm{H}}(\psi, p, h, \mathrm{H}(t)) = (1-p)\mathcal{L}[\psi(t,p) - \mathrm{q}_0(t)] - p\hbar \mathrm{H}(t)\mathrm{N}[\psi(t,p)] \tag{8}$$

where N is a general nonlinear operator; and $\mathrm{q}(t)$ is generally an unknown function of variable t which is time response of beam in this paper as mentioned in the previous section. Also h, $\mathrm{H}(t)$, ψ and $\mathcal{L}$ are non-zero auxiliary parameter, non-zero auxiliary function, a function of t and p and auxiliary linear operator respectively. As p increases from zero to one, $\psi(t,p)$ varies from the initial guess $(q_0(t))$ to the precise solution. It should be mentioned that there is not rigorous theory for choosing auxiliary linear operators [20]. Setting $\overline{\mathrm{H}}(\psi, p, h, \mathrm{H}(t)) = 0$ where p considered to be zero, leads to:

$$\psi(t,0) = \mathrm{q}_0(t) \tag{9}$$

The following initial conditions are considered:

$$\mathrm{q}_0(0) = a_0, \qquad \frac{\mathrm{d}\mathrm{q}_0(0)}{\mathrm{d}t} = 0 \tag{10}$$

The functions $\mathrm{q}(t)$ and Ω which are respectively time response and frequency of the beam, could be extracted as power series of p by using Taylor theory as follows:

$$\mathrm{q}(t) = (t,p) = \psi(t,0) + \sum_{m=1}^{+\infty} \frac{1}{m!} \frac{\partial^m \psi(t,p)}{\partial p^m}\Big|_{p=0} p^m = \mathrm{q}_0(t) + \sum_{m=1}^{+\infty} \mathrm{q}_m(t) p^m \tag{11}$$

$$\Omega = \Omega_0 + \sum_{m=1}^{+\infty} \frac{1}{m!} \frac{\partial^m \Omega(p)}{\partial p^m}\Big|_{p=0} p^m = \Omega_0 + \sum_{m=1}^{+\infty} \Omega_m p^m \tag{12}$$

where $\mathrm{q}_m(t)$ and Ω_m are called m-order transformations. The m-order approximation of $\mathrm{q}(t)$ is calculated as follows[15] :

$$\mathcal{L}[\mathrm{q}_m(t) - x_m \mathrm{q}_{m-1}(t)] = h\mathrm{H}(t)\mathrm{R}_m(\mathrm{q}_{m-1}, \Omega_{m-1}) \tag{13}$$

where x_m, q_{m-1}, Ω_{m-1} . Also $\mathrm{R}_m(\mathrm{q}_{m-1}, \Omega_{m-1})$ are expressed as:

$$x_m = \begin{cases} 0 & m = 1 \\ 1 & m > 1 \end{cases} \tag{14}$$

$$\mathrm{R}_m\left(\mathrm{q}_{m-1}, \Omega_{m-1}\right) = \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}\mathrm{N}\left[\psi(t,p), \Omega(p)\right]}{\mathrm{d}p^{m-1}}\Big|_{p=0} \tag{15}$$

$$\mathrm{q}_{m-1} = \left\{\mathrm{q}_1, \mathrm{q}_2, \mathrm{q}_3, \ldots, \mathrm{q}_{m-1}\right\} \tag{16}$$

$$\Omega_{m-1} = \left\{\Omega_0, \Omega_1, \Omega_2, \ldots, \Omega_{m-1}\right\} \tag{17}$$

where $\mathrm{N}[\psi(t,p), \Omega(p)]$ is nonlinear operator. Initial conditions are considered in the following form:

$$\mathrm{q}_m(0) = 0, \qquad \frac{\mathrm{dq}_m}{\mathrm{d}t}(0) = 0 \tag{18}$$

4. Homotopy Pade technique

The Pade approximant expands a function by a rational function of a given order precisely[18, 19]. The Pade technique accelerates convergence of given series. The Homotopy Pade method combines HAM with Pade technique. The Homotopy Pade method reduces the number of order of approximation required to get a precise answer. The [1,1] Homotopy Pade approximation for frequency and deflection is written as follows [25]:

$$\Omega[1,1]_{\text{pade}} = \frac{\Omega_1\Omega_0 + \Omega_1^{\ 2} - \Omega_2\Omega_0}{\Omega_1 - \Omega_2} \tag{19}$$

$$\mathrm{q}[1,1]_{\text{pade}} = \frac{\mathrm{q}_1\mathrm{q}_0 + \mathrm{q}_1^{\ 2} - \mathrm{q}_2\mathrm{q}_0}{\mathrm{q}_1 - \mathrm{q}_2} \tag{20}$$

5. Application of HAM

In this paper, nonlinear forced vibration of a clamped-clamped buckled beam under axial force and transverse harmonic load is presented using equation (3). This equation is nonhomogeneous; therefore its general solution ($\mathrm{q}(t)$) is obtained by the summation of the homogeneous solution ($\mathrm{q}_h(t)$) and the particular solution ($\mathrm{q}_p(t)$):

$$\mathrm{q}(t) = \mathrm{q}_h(t) + \mathrm{q}_p(t) \tag{21}$$

5.1.Homogeneous solution

Under the transformation $\tau = \Omega t$ the homogeneous solution of equation (3) becomes as follows:

$$\Omega^2\frac{d^2q}{d\tau^2}+\omega^2 q+\gamma q^3=0 \tag{22}$$

In order to obtain the answer of the above equation, the first guess of $q(t)$ which satisfies initial conditions is chosen as follows:

$$q_{0h}(\tau)=a_0\cos\tau \tag{23}$$

To create the Homotopy function, the linear operator could be expressed as:

$$\mathcal{L}[\phi(\tau,p)]=\Omega_0^{\ 2}(\frac{\partial^2\phi(\tau,p)}{\partial\tau^2}+\phi(\tau,p)) \tag{24}$$

The nonlinear operator could be written as:

$$N[\phi(\tau,p)]=\Omega^2(p)\frac{\partial^2\phi(\tau,p)}{\partial\tau^2}+\omega^2\phi(\tau,p)+\gamma\phi^3(\tau,p) \tag{25}$$

Since the solution of equation must comply with the general form of the base functions, the auxiliary function ($H(\tau)$) must be assumed as:

$$H(\tau)=1 \tag{26}$$

Therefore, with regard to equations (13) to (15), first order transformation equation could be written as:

$$\Omega_0^{\ 2}(\frac{\partial^2 q_{1h}(\tau)}{\partial\tau^2}+q_{1h}(\tau))=h(-a_0\Omega_0^{\ 2}\cos\tau+\omega^2 a_0\cos\tau+\gamma a_0^{\ 3}\cos^3(\tau)) \tag{27}$$

Solving above equation, ω_0 and q_{1h} are obtained as:

$$\Omega_0=\sqrt{\omega^2+\frac{3\gamma a_0^{\ 2}}{4}} \tag{28}$$

$$q_{1h}=b_1\cos\tau+b_2\sin\tau+D_2\cos 3\tau \tag{29}$$

Coefficients b_1, b_2 and D_2 are given in appendix A. The higher-order approximations is obtained similarly. Assuming $m = 2$ in equations (13) to (15) leads to the following result for second order approximation (q_{2h}):

$$q_{2h}=b_3\cos\tau+b_4\sin\tau+D_3\cos 3\tau+D_4\sin 3\tau+D_5\cos 5\tau \tag{30}$$

Coefficients b_3, b_4, D_3, D_4 and D_5 are given in appendix A. Consequently, From the coefficient of $\cos\tau$ in $R_2(q_{1h},\vec{\Omega}_1)$, Ω_1 is derived as:

$$\Omega_1=(-b_1\Omega_0^{\ 2}+b_1\omega^2+\frac{3}{4}D_2\gamma a_0^{\ 2}+\frac{9}{4}b_1\gamma a_0^{\ 2})/(2\Omega_0 a_0) \tag{31}$$

Also, from the coefficient of $\cos\tau$ in $R_3(q_{2h},\vec{\Omega}_2)$, Ω_2 could be derived as:

$$\Omega_2 = (-a_0\Omega_1^{\ 2} - 2\Omega_0\Omega_1 b_1 - \Omega_0^{\ 2} b_3 + \omega^2 b_3 + \frac{9}{4}\gamma a_0 b_1^{\ 2} + \frac{3}{4}\gamma a_0 b_2^{\ 2} + \frac{3}{2}\gamma a_0 D_2^{\ 2} + \frac{3}{2}\gamma a_0 b_1 D_2 + \frac{9}{4}\gamma b_3 a_0^{\ 2} + \frac{3}{4}\gamma D_3 a_0^{\ 2}) / (2a_0\Omega_0) \tag{32}$$

5.2. Particular solution

Assuming $\Gamma = kt$ in equation (3), the problem equation is rewritten as follows:

$$k^2 \frac{d^2 q}{d\Gamma^2} + \omega^2 q + \gamma q^3 = \tilde{f}\cos\Gamma \tag{33}$$

According to the type of transverse harmonic load ($\tilde{f}\cos\Gamma$), initial guess of problem particular solution could be written in the following form:

$$q_{0p}(\tau) = \pm b_0 \cos\Gamma \tag{34}$$

Linear operator ($\mathcal{L}$) for particular solution can be defined as:

$$\mathcal{L}[\phi(\tau,p)] = k^2 \left[\frac{\partial^2 \phi(\tau,p)}{\partial\Gamma^2} + \phi(\tau,p) \right] \tag{35}$$

Also, nonlinear operator could be written as:

$$N[\phi(\tau,p)] = k^2 \frac{\partial^2 \phi(\tau,p)}{\partial\Gamma^2} + \omega^2 \phi(\tau,p) + \gamma\phi^3(\tau,p) - \tilde{f}\cos\Gamma \tag{36}$$

Substituting equations (34), (35) and (36) in equation (13) while H and m are assumed to be one, leads to:

$$k^2 \left[\frac{\partial^2 q_{1p}(\tau)}{\partial\Gamma^2} + q_{1p}(\tau) \right] = h(-b_0 k^2 \cos\Gamma + \omega^2 b_0 \cos\Gamma + \gamma b_0^{\ 3} \cos^3(\Gamma) - \tilde{f}\cos\Gamma) \tag{37}$$

After some mathematical manipulations, the following solution is obtained for q_{1p}:

$$q_{1p} = D_6 \cos 3\Gamma \tag{38}$$

Coefficients D_6 is given in appendix A. Also from the coefficient of $\cos\Gamma$, b_0 is obtained from the following equation:

$$\left(\frac{3}{4}\gamma b_0^{\ 3} + \left(-k^2 + \omega^2\right) b_0 \right)^2 = \tilde{f}^{\ 2} \tag{39}$$

which is the equation of frequency response for HAM as mention in [43]. The higher-order approximations for private solution could be obtained in the same way. Substituting *m*=2 in equations (13) to (15) leads to:

$$k^2\left(\frac{\partial^2 \mathrm{q}_{2p}}{\partial \Gamma^2}+\mathrm{q}_{2p}\right)-k^2\left(\frac{\partial^2 \mathrm{q}_{1p}}{\partial \Gamma^2}+\mathrm{q}_{1p}\right)=hH\left(k^2\frac{\partial^3 \phi}{\partial \Gamma^2 \partial p}+\omega^2\frac{\partial \phi}{\partial p}+3\gamma\phi^2\frac{\partial \phi}{\partial p}+\tilde{f}\sin\Gamma\right)_{|p=0} \tag{40}$$

Solving equation (40) leads to the following solution for q_{2p}:

$$\mathrm{q}_{2p}=D_7\cos 3\Gamma+D_8\cos 5\Gamma \tag{41}$$

where D_7 and D_8 are given in appendix A. The frequency response for second order HAM is as follows [43].

$$\left(\frac{3}{4}\gamma b_0^{\,3}+\left(\omega^2-k^2\right)b_0-\frac{3c_0\gamma^2 b_0^{\,5}}{128\omega^2}\right)^2=f^{\,2} \tag{42}$$

where $c_0=-0.8$ is considered for softening behavior and $c_0=1$ is valid for hardening behavior [43]. According to equation (11), total solution for $\mathrm{q}(\tau)$ is derived as follows:

$$\mathrm{q}(\tau)=\mathrm{q}_0(\tau)+p^1\mathrm{q}_1(\tau)+p^2\mathrm{q}_2(\tau)+\ldots \tag{43}$$

where

$$\mathrm{q}_i(\tau)=\mathrm{q}_{hi}(\tau)+\mathrm{q}_{pi}(\tau), i=0,1,2,\ldots \tag{44}$$

Equation (12) yields to following result for natural frequency ($\Omega(p)$):

$$\Omega=\Omega_0+p^1\Omega_1+p^2\Omega_2+\ldots \tag{45}$$

where

$$\Omega_i=\Omega_{hi}+\Omega_{pi}, i=0,1,2,\ldots \tag{46}$$

6. Results and discussions

In order to show the accuracy and effectiveness of HAM and Homotopy Pade technique, in this section some results are obtained for nonlinear vibration of the clamped-clamped buckled beam subjected to axial force and transverse harmonic load. The values of beam characteristics are given in Table 1 (unless mentioned otherwise).

Table 1. The values of beam parameters

Width(mm)	Thickness(mm)	Length(mm)	E(GPa)	$\rho(kg/m^3)$	$\hat{a}$
3.9	6.4	485	190	7880	1

Fig. 2 shows analytic response of first and second orders HAM for different values of the auxiliary parameter h in a given point where $t=1, \mathrm{P}=0$ and $w_{max}=0.5$mm. Substituting $\mathrm{P}=0$ and the values of Table 1 in equations (5) and (6) leads to $\omega=782.0385$ and $\alpha_3=3.2974\times 10^{10}$. Then analytical results are compared with Runge Kutta method solution obtained for the same values of the parameters. It is observed that analytic solution for $h=-1$ confirms to Runge Kutta method solution. Also some researchers have used h=-1 [9, 25].

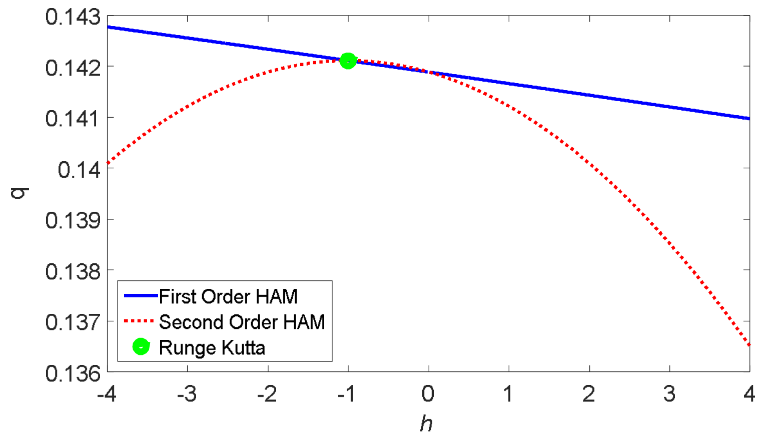


Fig.2. Curve of q versus h obtained with HAM and Runge Kutta solution.

Table 2. Comparison of nonlinear frequencies obtained from first order HAM, second order HAM, homotopy pade and Runge Kutta method solution with experimental results [44] for different values of l and $w_{\max}$.

l (mm)	$w_{\max}$ (mm)	second order HAM(Hz)	First order HAM (Hz)	[1,1]homotopy pade (Hz)	Runge Kutta (Hz)	Experimental[44] (Hz)
485	4	151.01	151.01	151.01	155.42	150
485	3	145.21	145.21	145.21	145.84	142.8
485	2	140.89	140.89	140.89	139.75	140
485	1	138.23	138.23	138.23	138.46	137.5
416	3.5	201.08	201.08	201.08	201.60	199.9
416	3	197.37	197.37	197.37	194.37	197.1
416	2	191.51	191.51	191.51	193.09	191.9
416	1	187.89	187.89	187.89	187.48	188.6
326	2.5	316.18	316.18	316.18	316.51	316.6
326	1.5	308.42	308.42	308.42	306.96	308.3
326	1	305.95	305.95	305.95	305.62	306.6
326	0.5	304.46	304.46	304.46	304.44	304.6

Table 2 compares the obtained results for nonlinear frequency via first order HAM, second order HAM, [1,1] Homotopy Pade method and Runge Kutta method with experimental results Rezaee *et al.* [44] for $P = 0N$ and different values of l and $w_{\max}$. It is shown in Table 2 that there are good agreement between the results of HAM , [1,1] Homotopy pade method and experimental results. It is concluded that the convergence acceleration of the HAM solution is impressive and the first order approximation of HAM is accurate. In addition, it is concluded that the Runge Kutta method accuracy is less than mentioned approximate analytical methods. Also, Table 3 compares the results of frequency tatio (Ω/ω) of first order HAM, second order HAM, [1,1] Homotopy Pade technique and Runge Kutta method with the results obtained by [2, 8] for $P =$

$0N$ and different values of $\frac{w_{max}}{r}$. Similar to the results of Table 2, it is obvious from Table 3 that the HAM convergence is fast and first order HAM is very accurate. Table 3 demonstrates an excellent agreement between the results of HAM ,[1,1] Homotopy Pade and the results obtained by[2, 8]. The Runge Kutta method results difference from literature results is greater than HAM and [1,1] homotopy pade method results difference from literature. As verified in this case study, HAM and Homotopy Pade method provide precise answers which are valid for a wide range of initial deflection and beam characteristic.

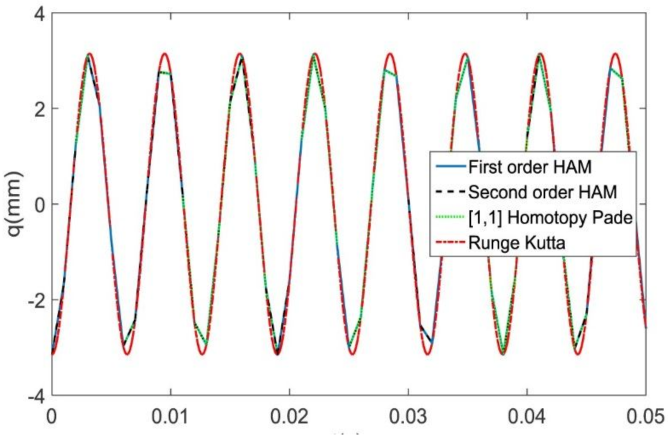
Table 3. Comparison of frequency ratio (Ω/ω) obtained from first order HAM, second order HAM, homotopy pade and Runge Kutta method solution with results obtained by[2, 8] for different values of $\frac{w_{max}}{r}$.

$\frac{w_{max}}{r}$	Second order HAM	First order HAM	[1,1]Homotopy Pade	Runge Kutta	Azrar et al. (1999)	Quasi, 1993
1	1.02219	1.02219	1.02219	1.01496	1.02219	1.0628
1.5	1.04917	1.04917	1.04917	1.04478	1.04917	1.1322
2	1.08566	1.08566	1.08567	1.07122	1.08567	1.2140
2.5	1.13065	1.13066	1.13066	1.17072	1.13066	1.3017
3	1.18309	1.183096	1.18310	1.21229	1.18310	1.3904
3.5	1.24198	1.24198	1.24199	1.28145	1.24199	1.4786
4	1.30640	1.30641	1.30640	1.21637	1.30640	1.5635
4.5	1.37557	1.37559	1.37556	1.27895	1.37556	1.6418
5	1.44880	1.44882	1.44876	1.323843	1.44876	-

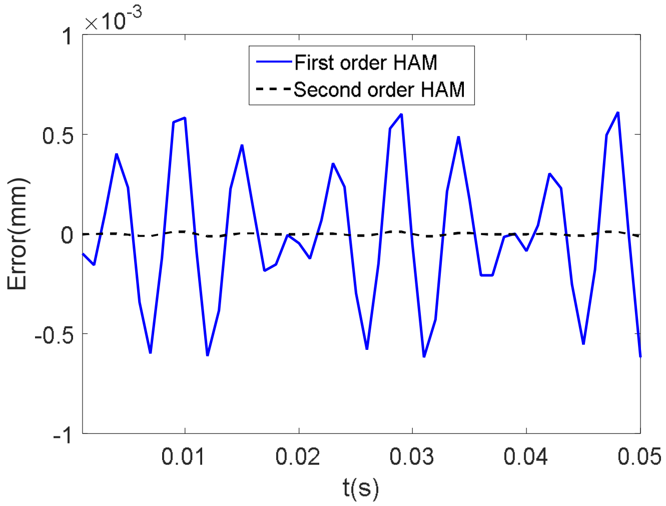
Fig. 3a shows time response of beam nonlinear free vibration via first order HAM, second order HAM, [1,1] Homotopy Pade method and Runge Kutta method for $w_{\max} = 5\text{mm}, f = 0$ and $P = 0$. Fig. 3b depicts the error of first and second orders HAM response from Homotopy Pade method response. It can be seen that although the error of both first and second orders HAM is small, second order HAM error is smaller. Fig. 3c demonstrates the error of Runge Kutta method response from Homotopy Pade method response. Comparison of Figs. 3b and c demonstrates that the error of Runge Kutta method response is greater than the error of first and second orders HAM.

Fig. 4a illustrates time response of beam nonlinear forced vibration via first order HAM, second order HAM, Homotopy Pade method and Runge Kutta method for $w_{max} = 5mm, P = 500, f = 1000\ and\ k = 700\text{rad/s}$. Fig. 4b shows the error of first and second orders HAM response from Homotopy Pade method response. It can be observed from Fig. 4b that the error of response obtained by second order HAM is smaller than the error of first order HAM. Fig. 4c depicts the error of Runge Kutta method response from Homotopy Pade method response. Figs. 4a and c demonstrate that the Runge Kutta method response error is great.

a.



b.



c.

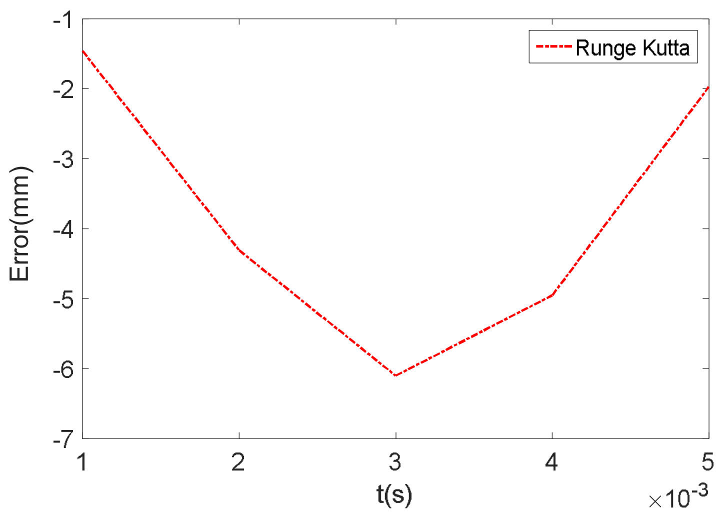
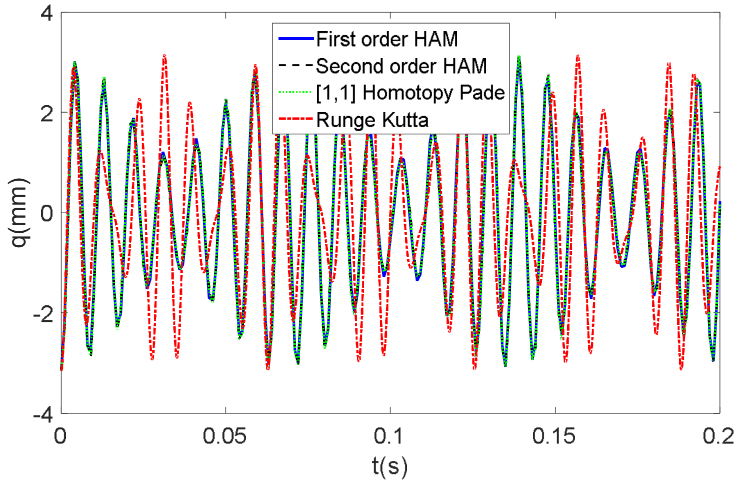
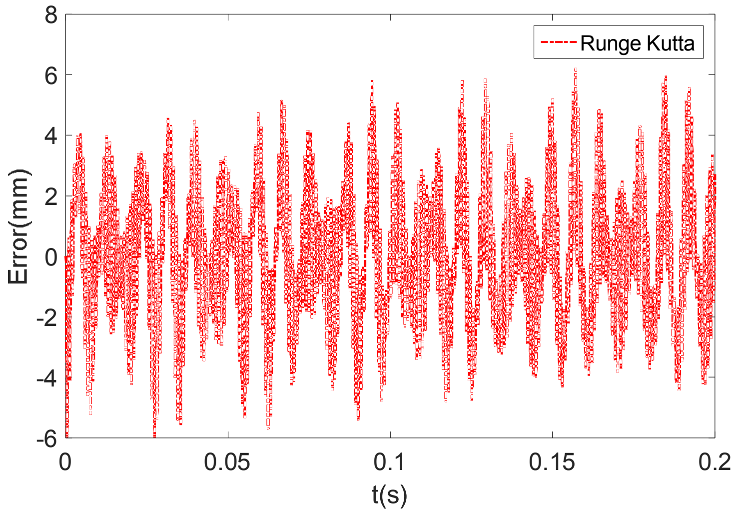


Fig.3. Time response (q) for free vibration of the beam where $w_{\max} = 5\text{mm}, f = 0$ and $P = 0$.

a.



b.



c.

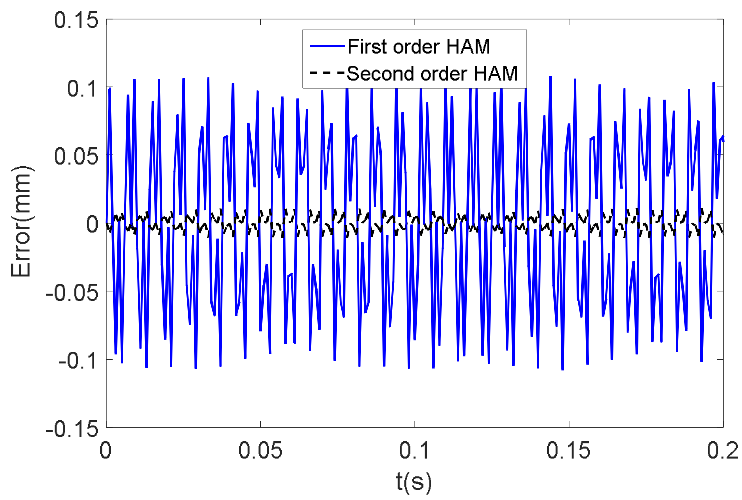
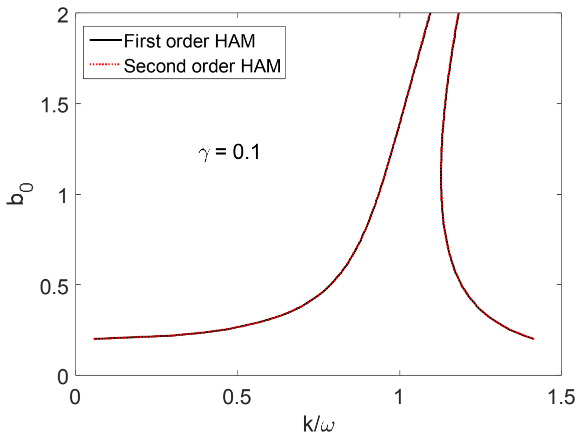


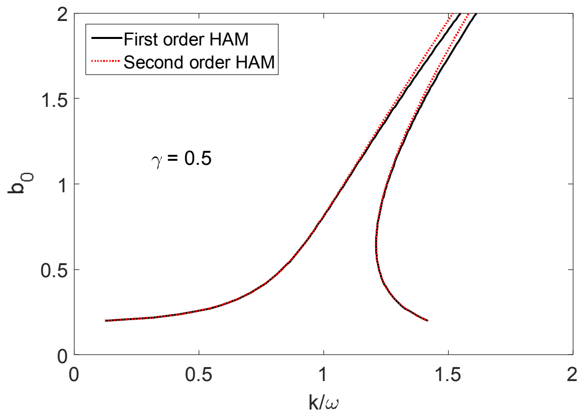
Fig.4. Time response (q) for forced vibration of the beam where $w_{max} = 5mm, P = 500, f = 1000 \; and \; k = 700\text{rad/s}$.

The results show that HAM method is better than Runge Kutta method in obtaining nonlinear response of the system. Now, in the next step, we depict Fig. 5 to obtain frequency response by first and second orders HAM for $\omega = 1, \tilde{f} = 0.2$ and different positive values of nonlinear parameter (γ). As seen from Fig. 5 the increase of nonlinear parameter (γ) leads to increase in the hardening behavior of the system. Also, increasing of nonlinear parameter (γ), leads to increase in the difference of first order HAM response from second order HAM response. Fig. 6 shows frequency response by first and second orders HAM for $\omega = 1, \tilde{f} = 0.2$ and different negative values of nonlinear parameter (γ). It is concluded from Fig. 6 that the increase of absolute value of nonlinear parameter (γ) leads to increase in the system softening behavior. Also as the absolute value of γ increases, the difference of first order HAM response from second order HAM response increases.

a.



b.



c.

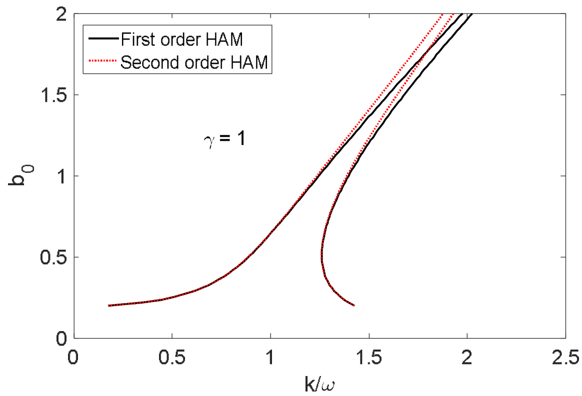


Fig.5. Frequency response for $\omega = 1, \tilde{f} = 0.2$ and different positive values of nonlinear parameter (γ).

a.

b.

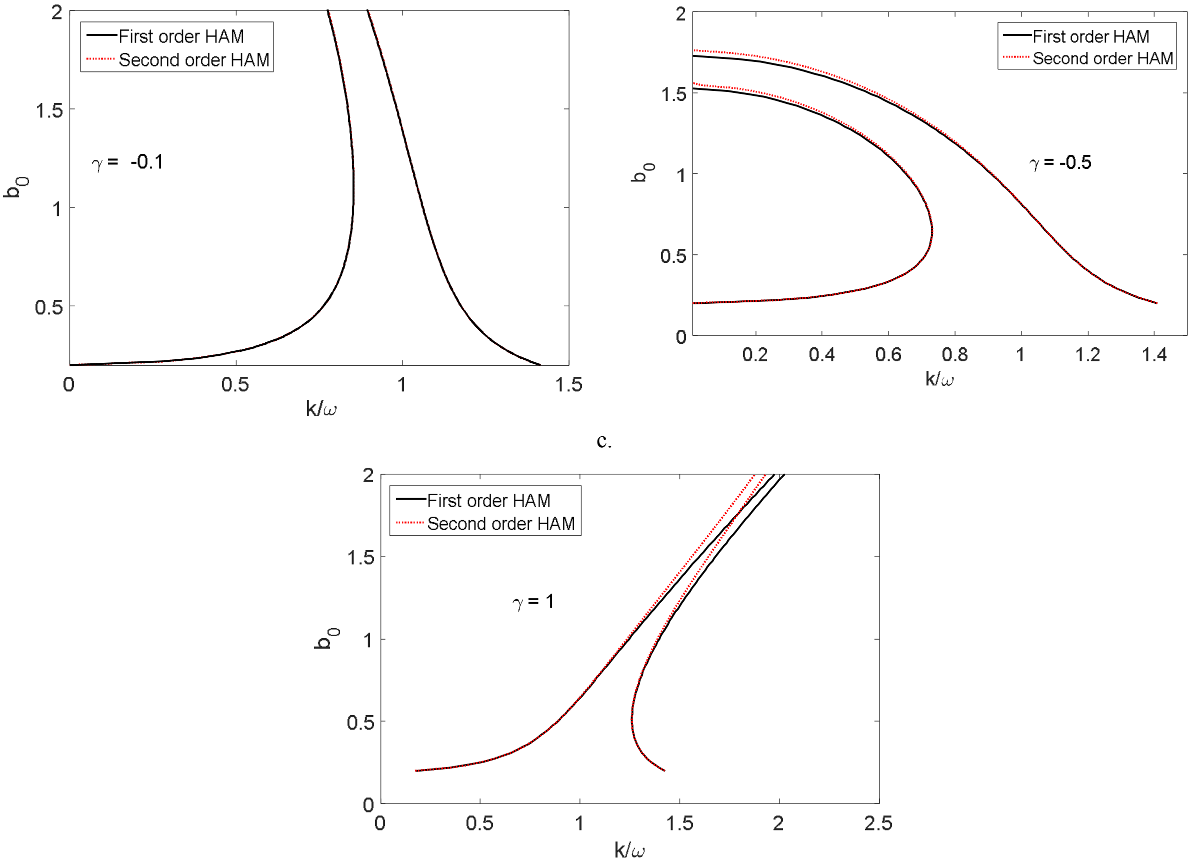


Fig.6. Frequency response for $\omega = 1, \tilde{f} = 0.2$ and different negative values of nonlinear parameter (γ).

7. Conclusion

Analytical solutions are used as a reference frame for the validation and verification of the numerical approaches. Also, the analytical solutions present an insight and thought comprehension of the effect of the system parameters and initial conditions. Consequently, in this paper, for the first time the HAM and Homotopy Pade method have been employed to study nonlinear forced vibration of Euler-Bernoulli clamped-clamped buckled beam subjected to axial force and transverse harmonic load. Analytical and numerical solutions for the nonlinear frequency for different values of initial deflection and beam length are obtained and compared with experimental results of literature. In addition, frequency ratio is obtained for different values of beam initial deflection and compared with literature. Also time response of free and forced vibration of beam through HAM, Homotopy pade method and Runge Kutta method has been obtained. In addition, frequency response of beam through HAM has been drawn. It can be concluded that HAM method answers converge quickly and its components are simply computed. Also, the HAM and Homotopy Pade method have excellent accuracy for wide range of initial deflection and beam characteristic. It is concluded that HAM and Homotopy Pade method results require small computational effort and only the second order approximation of HAM or [1,1] Homotopy Pade method leads to precise answers. However, further study is required to better discover the effect of different parameters on the accuracy of HAM and Homotopy Pade method.

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Appendix A

By applying initial conditions to equation (43) the coefficients have been obtained as follows:

$$D_2 = -\frac{h\gamma a_0^{\ 3}}{32\Omega_0^{\ 2}} \tag{A.1}$$

$$D_3 = -\frac{h}{8\Omega_0^{\ 2}}(\mathrm{D}_2\,\omega^2 - 9\Omega_0^{\ 2}D_2 + \frac{3}{4}b_1\gamma a_0^{\ 2} + \frac{3}{2}D_2\gamma a_0^{\ 2} - 8D_2\frac{\Omega_0^{\ 2}}{h}) \tag{A.2}$$

$$D_4 = -\frac{3h}{32\Omega_0^{\ 2}}b_2\gamma a_0^{\ 2} \tag{A.3}$$

$$D_5 = -\frac{3h}{96\Omega_0^{\ 2}} D_2 \gamma a_0^{\ 2} \tag{A.4}$$

$$D_6 = -\frac{h\gamma b_0^{\ 3}}{32k^2} \tag{A.5}$$

$$D_7 = -\frac{h}{8k^2}(-9k^2 + \omega^2 + \frac{3}{2}\gamma b_0^{\ 2} - 8\frac{k^2}{h})D_6 \tag{A.6}$$

$$D_8 = -\frac{h\gamma b_0^{\ 2} D_6}{32k^2} \tag{A.7}$$

$$b_1 = \left(-D_2 - b_0 - D_6\right) \tag{A.8}$$

$$b_2 = 0 \tag{A.9}$$

$$b_3 = -(D_3 + D_5 + D_7 + D_8) \tag{A.10}$$

$$b_4 = -3D_4 \tag{A.11}$$