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Journal of Theoretical and Applied Vibration and Acoustics

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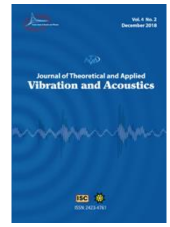

Nonlinear vibration analysis of axially moving strings in thermal environment

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ARTICLE INFO

Article history:
Received 31 July 2018

Received in revised form 28 November 2018

Accepted 08 December 2018

Available online 14 December 2018

Keywords:
Axially moving string,
Nonlinear vibration,
Acoustics,
Multiple time scales method,
Critical speed.

ABSTRACT

In this study, nonlinear vibration of axially moving strings in thermal environment is investigated. The vibration characteristics of the system such as natural frequencies, time domain response and stability states are studied at different temperatures. The velocity of the axial movement is assumed to be constant with minor harmonic variations. It is presumed that the system and the environment are in thermal equilibrium. Using Hamilton's principle, the system equation of motion, and t[†]he boundary conditions are derived and then solved by applying Multiple Time Scales (MTS) method. The effect of temperature on the vibration characteristics of the system such as linear and nonlinear natural frequencies, stability, and critical speeds is investigated. Considering ideal and non-ideal boundary conditions for the supports, nonlinear vibration of the system is discussed for three different excitation frequencies. The bifurcation diagrams for ideal and non-ideal boundary conditions are presented under the influence of temperature at various speeds.

1. Introduction

To study vibration characteristics in some engineering equipment such as power transmission belts, tape drives, winding fibers, lift string, cranes and elevators, magnetic tapes, shearing strings, etc., they may be modeled as a string. Speed is an important factor in systems with an axially moving component. On the other hand, vibration plays a major role in limiting the

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http://dx.doi.org/10.22064/tava.2018.91055.1114

velocity of movement of these structures. Vibrations of axially moving structures have been investigated by many researchers. One of the first studies in this regard is the research of Sack [1] in which a string passed through two ideal pulleys with constant speed. Among the early studies devoted to the linear vibration analysis of axially moving strings is the study of Archibald and Emslie [2] where the string was assumed to be uniform and moving with a constant speed. Using Hamilton’s principle, Miranker [3] has extracted a vibrational model of a tape moving between a pair of pulleys. An approximate solution for vibration of an axially moving string has been obtained by Mote[4]. In this study, by assuming a variable speed for the string, stability of the system has been investigated.

Vibrations and the stability of the saw blade have been investigated by Ulsoy and Mote [5]. Pakdemirli *et al*.[6] investigated transverse vibrations of the axially accelerating string. The equation of motion was derived using the Hamilton's principle and then the governing differential equation which was obtained by Galerkin method was solved. Results show that instability occurs in some amplitudes and frequencies related to harmonic axial velocity.

The effect of damping on the linear vibration of axial moving structures has been addressed in several studies. Many studies such as (Mote; Ulsoy and Mote) [4, 5] that were performed regarding the axially moving string with damping has shown that the natural frequencies, vibration amplitudes and stability of the system depend on the damping coefficient and the speed of motion.

The first investigation in which the dynamic response of axially moving viscoelastic string has been studied is the study of Fung *et al*.[7]. In another study, the qualitative aspect of parametric excitation due to the variable velocity of viscoelastic string has been investigated by the same authors (Fung *et al*) [8].

Chen and Chen[9] have analyzed the nonlinear vibration of accelerated viscoelastic string with an approximate analytical approach. The numerical results have shown the effect of material properties, velocity, and initial tensile strength on the steady state response. Ghayesh [10] has investigated the nonlinear vibration of an axially moving viscoelastic string supported by a viscoelastic guide. In this study, the velocity is considered as a constant value with small harmonic oscillations. The viscoelastic material has been modeled as a combination of parallel spring and damper. Numerical simulations have demonstrated the effect of different factors such as axial speed, string length, damping coefficient and hardness on the stability and natural frequency. Mockensturm and Guo [11] have studied the nonlinear vibrations of an axially moving viscoelastic string under parametric excitation. The bifurcation phenomenon and chaos of axially moving viscoelastic string have been investigated by Chen *et al*.[12]. According to the numerical simulations, periodic, quasi-periodic, and chaotic motions can occur in the transverse vibrations of the axially moving viscoelastic strings. The dynamic response and stability of a viscoelastic belt under parametric excitation have been investigated by Zhang and Zu [13]. The linear viscoelastic differential constitutive law has been employed to characterize the material property of the belts. In this paper, generalized equation of motion has been obtained for a viscoelastic moving belt with geometric nonlinearity. The effects of viscoelastic parameters, excitation frequencies and amplitudes, and axial moving speeds on the dynamic response have been investigated. Chen *et al*.[14] have studied the nonlinear dynamics of transverse motion of axially moving strings. The governing equation was derived using the Newton's second law for two longitudinal and transverse directions, and was then solved using Galerkin's method (Chen *et al*) [14].

Lepidi and Gattulli[15] studied temperature effects on the static and dynamic response of suspended inclined cables. The sensitivity of the linear frequencies to temperature changes is discussed. Yurddas *et al*.[16] used perturbation techniques to investigate nonlinear vibration of axially moving strings with multi-nonideal supports. Vibration has been studied for three different velocity variation frequencies. Marynowski and Kapitaniak [17] studied dynamics of axially moving systems which includes some results of studies in the field of dynamics of axially moving viscoelastic systems. Malookani and van Horssen [18] employed the two timescales perturbation method and the Laplace transform method to solve an initial-boundary value problem for a linear-homogeneous equation describing an axially moving string. Yang *et al*. [19] analyzed nonlinear vibration of axially moving strings based on gyroscopic complex modes. The method of multiple scales is used to investigate 1:3 internal resonance case.

Although vibration analysis of some structures under thermal load has been considered by researchers, to the knowledge of the authors, vibrations of the axially moving strings in thermal environment have not been studied so far. In this study, the nonlinear vibration of an axially moving string in thermal environment has been investigated.

2. Equation of motion

A uniform string with cross-sectional area A, density ρ, length l which is under the initial tensile force of P is assumed to move between two pulleys. The string is moving at a speed of v. The physical model of the system is shown in Fig. 1 where, $w(x,t)$ denotes the transverse displacement of the string. The system is under thermal load such that the system and the environment are in thermal equilibrium and the temperature of the system is constant. The string is assumed to move at a constant speed with harmonic minor variations. Moreover, it is assumed that the initial tensile force and temperature gradient are such that string remains elastic and its specifications do not change. In this study, the nonlinear vibration of the system is investigated only in transverse direction.

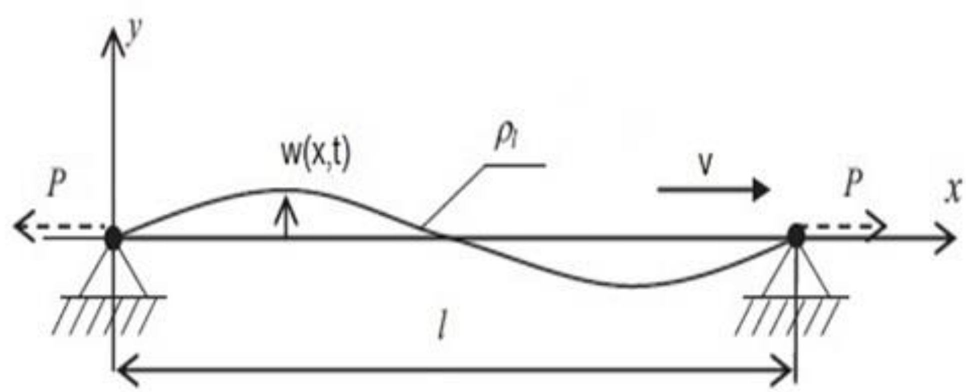

Fig 1: Axially moving string model

The nonlinear equation of motion is obtained by using Hamilton's principle given in Eq. 1.

$$\delta\int_{t_1}^{t_2}\left(T-U-W_{nc}\right)dt=0 \tag{1}$$

where T, U and W_{nc} are kinetic energy, potential energy and work of non-conservative forces, respectively.

The total kinetic energy of the system may be written as:

$$T=\int_0^l \rho A\left(v^2+v_y{}^2\right)dx \tag{2}$$

where $v_y=\frac{Dw}{Dt}=w_t+vw_x$. The total potential energy of the system considering temperature change is:

$$U=\int_0^l\{P\left(\frac{1}{2}w_x{}^2-\alpha\Delta T\right)+\frac{1}{2}EA\left(\frac{1}{2}w_x{}^2-\alpha\Delta T\right)^2\}dx \tag{3}$$

where $w_x=\frac{\partial w}{\partial x}$ and $w_t=\frac{\partial w}{\partial t}$. In Eq. 3, α and E are thermal expansion coefficient and elastic modulus of string, respectively, and ΔT is temperature difference. By substituting Eqs. 2 and 3 into Eq. 1, nonlinear equation of motion is obtained as:

$$\rho A\left(w_{tt}+\dot{v}w_x\right)+2\rho Avw_{xt}+\left(\rho Av^2-P+EA\alpha\Delta T\right)w_{xx}-EA\left(\frac{3}{2}w_x{}^2\right)w_{xx}=0 \tag{4}$$

For non-ideal boundary conditions we have (Yurddas et al. (2013)):

$$w(0,t)=\varepsilon\sqrt{\varepsilon}\mathrm{a}\ ,\ w(l,t)=\varepsilon\sqrt{\varepsilon}b \tag{5}$$

The values of a and b are in the order of 1, and ε is usually much smaller than 1 which show small displacements in the boundaries. In order to obtain the results independent of the geometry and material properties, Eq. 4 is rewritten in the dimensionless form as follows:

$$W_{\tau\tau}+\frac{\partial V}{\partial\tau}W_z+2VW_{z\tau}+\left(V^2-1+\alpha_T\right)W_{zz}-GW_z^2W_{zz}=0 \tag{6}$$

where the parameter G and non-dimensional terms are defined as:

$$G=\frac{3EA}{2P}$$
$$W=\frac{w}{L}\ ,\ z=\frac{x}{L},\quad \tau=t\sqrt{\frac{P}{\rho AL^2}},\quad V=v\sqrt{\frac{\rho A}{P}},\quad \alpha_T=\frac{EA\alpha\Delta T}{P}\ ,\ a_0=\frac{a}{L},\quad b_0=\frac{b}{L} \tag{7}$$

The dimensionless velocity V slightly deviates from a constant V_0 based on the following equation in which the amplitude and frequency of the variation are shown by εV_1 and Ω, respectively.

$$V = V_0 + \varepsilon V_1 sin\left(\Omega\tau\right) \quad (8)$$

where ε is a parameter much smaller than one. By substituting Eq. 8 into Eq. 6, and assuming $W = \sqrt{\varepsilon u}$, the equation of motion and boundary conditions become

$$u_{\tau\tau} + \varepsilon V_1\Omega\cos\left(\Omega\tau\right)u_z + 2\left(V_0 + \varepsilon V_1 \sin\left(\Omega\tau\right) + u_{\tau z}\right) + (V_0^2 + 2\varepsilon V_0 V_1 sin\left(\Omega\tau\right)$$
$$+\varepsilon^2 V_1^2 sin\left(\Omega\tau\right) - 1 + \alpha_T)u_{zz} - \varepsilon G u_{zz} u_z^2 = 0 \quad (9)$$

$$u\left(0,\tau\right) = \varepsilon \mathrm{a}_0 \ , \ u\left(1,\tau\right) = \varepsilon b_0 \quad (10)$$

3. Multiple time scales method

To solve the nonlinear differential equation obtained in Section 2, MTS method is employed. First, $u\left(z,\tau\right)$ is expanded as (Nayfeh and Mook)[20]:

$$u\left(z,\tau\right) = u_0\left(z, T_0, T_1\right) + \varepsilon u_1\left(z, T_0, T_1\right) + ... \quad (11)$$

Here, u_0 and u_1 are special functions at order 1 and ε, respectively. $T_0 = \tau$ is a fast time scale and $T_1 = \varepsilon\tau$ is a slow time scale. As a result, time and space derivatives are obtained as:

$$u_z = \frac{\partial u_0}{\partial z} + \varepsilon \frac{\partial u_1}{\partial z} = u_0^{'} + \varepsilon u_1^{'} + ... \quad (12)$$

$$\frac{\partial}{\partial \tau} = D_0 + \varepsilon D_1 \quad (13)$$

$$\frac{\partial^2}{\partial \tau^2} = D_0^2 + 2\varepsilon D_0 D_1 + \dots \quad (14)$$

$$D_n = \frac{\partial}{\partial T_n} \quad (15)$$

Substitute Eqs. 11-15 into Eq. 9, one obtains:

$$\left(D_0^2 + 2\varepsilon D_0 D_1 + \dots\right)\left(u_0 + \varepsilon u_1\right) + \varepsilon V_1 \Omega\cos\left(\Omega T_0\right)\left(u_0^{'} + \varepsilon u_1^{'}\right)$$
$$+2\left(V_0 + \varepsilon V_1 \sin\left(\Omega T_0\right)\right)\left(D_0 + \varepsilon D_1\right)\left(u_0^{'} + \varepsilon u_1^{'}\right)$$

$$+\left(V_0^2+2\varepsilon V_0 V_1 sin\left(\Omega T_0\right)+\varepsilon^2 V_1^2 sin\left(\Omega T_0\right)-1+\alpha_T\right)\left(u_0^{''}+\varepsilon u_1^{''}\right)$$
$$-\varepsilon G\left(u_0^{''}+\varepsilon u_1^{''}\right)(u_0^{'}+\varepsilon u_1^{'})^2=0 \quad (16)$$

To solve the problem in hand, different orders of ε are separated as follows.

3.1. Linear problem (order 1)

The equation obtained from the first order of Eq. 16 which is a linear problem is:

$$D_0^2 u_0+2V_0 D_0 u_0^{'}+\left(V_0^2-1+\alpha_T\right)u_0^{''}=0 \quad (17)$$

To solve the linear problem, the special function of order 1 is considered as:

$$u_0=A\left(T_1\right)e^{i\omega_n T_0}U_0\left(z\right)+\bar{A}\left(T_1\right)e^{-i\omega_n T_0}\bar{U}_0\left(z\right) \quad (18)$$

By using special function in Eq. 17, the following equation and boundary conditions are obtained:

$$-\omega_n^2 U_0+2iV_0\omega_n U_0^{'}+\left(V_0^2-1+\alpha_T\right)U_0^{''}=0$$
$$U_0\left(0\right)=0 \quad , \quad U_0\left(1\right)=0 \quad (19)$$

The solution of Eq. 19 in general form is (Yurddas et al. (2013)):

$$U_0=C_1e^{B_1 z}+C_2e^{B_2 z} \quad (20)$$

By substituting Eq. 20 into Eq. 19 and considering the boundary conditions, the linear natural frequencies and mode shapes functions are obtained as:

$$\omega_n=\frac{1-\alpha_T-V_0^2}{\sqrt{1-\alpha_T}}n\pi \quad (21)$$

$$U_0=C_1e^{i\frac{n\pi V_0 z}{\sqrt{1-\alpha_T}}}\sin\left(n\pi z\right) \quad (22)$$

For different temperatures and velocities, the first four natural frequencies are given in Table 1. The first natural frequency versus axial velocity diagram is presented Fig. 2. According to the results, natural frequencies decrease when temperature increases. Also, the natural frequencies decrease by increasing the axial velocity. When the velocity is close to the critical velocity value, the natural frequency will reach zero. This point resembles the start of instability in the system, in which case the amplitude will increase significantly. According to Fig. 2, the critical velocity of the system which is a function of temperature, decreases if the temperature increases.

Table 1: First four natural frequencies for different values of temperature and velocity

α_T	V_0	ω_{n1}	ω_{n2}	ω_{n3}	ω_{n4}
-0.3	0	3.5819	7.1639	10.7459	14.3278
	0.2	3.4717	6.9435	10.4152	13.8870
	0.4	3.1411	6.2822	9.4233	12.5644
	0.6	2.5900	5.1800	7.7701	10.3601
0	0	3.1415	6.2831	9.4247	12.5663
	0.2	3.0159	6.0318	9.0477	12.0637
	0.4	2.6389	5.2778	7.9168	10.5557
	0.6	2.0106	4.0212	6.0318	8.04247
0.3	0	2.6284	5.2568	7.8853	10.5137
	0.2	2.4782	4.9564	7.4347	9.9130
	0.4	2.0276	4.0553	6.0829	8.1106
	0.6	1.2766	2.5533	3.8300	5.1067
0.6	0	1.9869	3.9738	5.9607	7.9476
	0.2	1.7882	3.5764	5.3646	7.1529
	0.4	1.1921	2.3843	3.5764	4.7686
	0.6	0.1986	0.3973	0.5961	0.7947

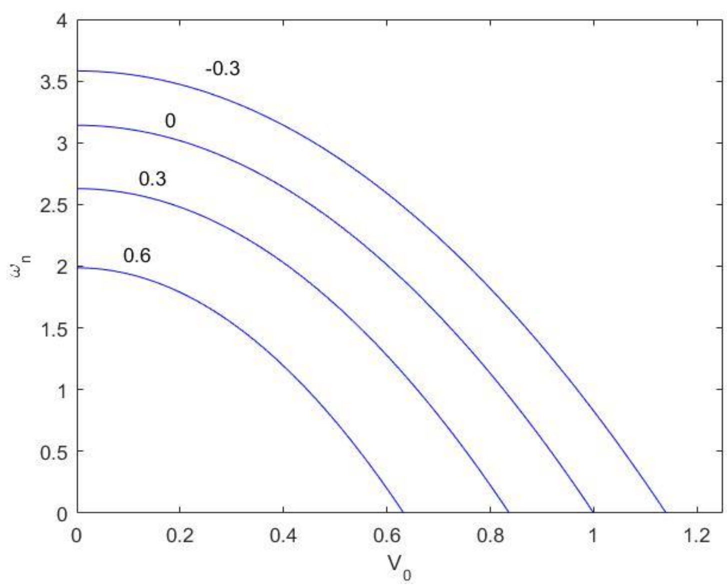

Fig 2: First natural frequency in terms of velocity for various temperatures α_T

The critical velocity for different temperatures can be obtained as $V_c = \sqrt{1-\alpha_T}$. For $\alpha_T = 1$, the critical velocity will be zero which means that the system will be unstable at any speed.

3.2. Nonlinear problem (order ε)

The equation of order ε is:

$$\begin{gathered} D_0^2 u_1 + 2V_0 D_0 u_1^{'} + \left(V_0^2 - 1 + \alpha_T\right) u_1^{''} = -2D_0 D_1 u_0 - 2V_0 D_0 u_0^{'} \\ -2V_1 \sin\left(\Omega T_0\right) D_0 u_0^{'} - V_1 \Omega \cos(\Omega T_0) u_0^{'} - 2V_0 V_1 sin\ (\Omega T_0)\ u_0^{''} + G u_0^{''} u_0^{'2} \end{gathered} \tag{23}$$

It can be seen that nonlinear terms appear and the response which can be divided into secular and non-secular parts is:

$$u_1\left(z, T_0, T_1\right) = \varphi\left(z, T_1\right) e^{i\omega_n T_0} + \psi\left(z, T_0, T_1\right) + cc \tag{24}$$

where φ and ψ correspond to the secular and non-secular terms of the equation, respectively, and cc denotes complex conjugate of the preceding terms. By using the following relations

$$\cos \Omega T_0 = \frac{1}{2}\left(e^{i\Omega T_0} + e^{-i\Omega T_0}\right) \ , \ \sin \Omega T_0 = \frac{1}{2i}\left(e^{i\Omega T_0} - e^{-i\Omega T_0}\right) \tag{25}$$

and substituting Eqs. 24 and 25 into Eq. 23, one obtains:

$$\begin{gathered} e^{i\omega_n T_0}\left[-\omega_n^2 \varphi + 2iV_0 \omega_n \varphi' + \left(V_0^2 - 1 + \alpha_T\right)\varphi''\right] \\ = -2\left[i\omega_n U_0 + V_0 U_0^{'}\right] D_1 A e^{i\omega_n T_0} + V_1\left(-\omega_n U_0^{'} - \frac{\Omega}{2} U_0^{'} + iV_0 U_0^{''}\right) A e^{i(\Omega+\omega_n)T_0} \\ + V_1\left(\omega_n \bar{U}_0^{'} - \frac{\Omega}{2}\bar{U}_0^{'} + iV_0 \bar{U}_0^{''}\right)\bar{A} e^{i(\Omega-\omega_n)T_0} \\ + G\left(\bar{U}_0^{''} U_0^{'2} + 2\bar{U}_0^{''} U_0^{'} \bar{U}_0^{'}\right) A^2 \bar{A} e^{i\omega_n T_0} + G\bar{U}_0^{''} U_0^{'2} A^3 e^{i3\omega_n T_0} + N.S.T + cc \end{gathered} \tag{26}$$

where N.S.T represents non-secular terms. Equation 26 is a non-homogeneous ordinary differential equation. If a nontrivial solution exists for the homogeneous part of the problem, the non-homogeneous part will have solution if it satisfies the solvability conditions. The non-ideal boundary conditions are defined as:

$$\begin{cases} \varphi\left(0, T_1\right) = a_0 A \\ \varphi\left(1, T_1\right) = b_0 A \end{cases} \tag{27}$$

The stability of the system for different ranges of Ω is investigated in the following. The results are obtained for $a_0 = 0$.

3.2.1. Ω far from zero and $2\omega_n$

If Ω is far from zero and $2\omega_n$, according to the non-homogenous equation and the boundary conditions, solution must satisfy the following solvability conditions:

$$D_1 A - K_1 A^2 \bar{A} + K_2 A = 0 \tag{28}$$

where K_1 and K_2 are defined as:

$$K_1 = \frac{\int_0^1 G\left(U_0 \bar{U}_0^{''} U_0^{'2} + 2\bar{U}_0 U_0^{''} U_0^{'} \bar{U}_0^{'}\right) dz}{\int_0^1 -2\left[i\omega_n \bar{U}_0 U_0 + V_0 \bar{U}_0 U_0^{'}\right] dz} = K_{1I} i \tag{29}$$

$$K_2 = \frac{\left(-b_0 e^{-i\frac{n\pi V_0}{\sqrt{1-\alpha_T}}} \cos n\pi + a_0\right) n\pi \left(V_0^2 - 1 + \alpha_T\right)}{\int_0^1 2\left[i\omega_n \bar{U}_0 U_0 + V_0 \bar{U}_0 U_0^{'}\right] dz} = K_{2R} + iK_{2I} \tag{30}$$

In order to solve Eq. 28, A is assumed as $A = \frac{1}{2} a_n e^{i\theta_n}$ and then the real and imaginary parts of the obtained equation is separated as:

$$\begin{cases} D_1 a_n = -K_{2R} a_0 \\ a_n D_1 \theta_n = \frac{1}{4} K_{1I} a_n^3 - K_{2I} a_n \end{cases} \tag{31}$$

By integrating Eq. 31, one obtains:

$$\begin{cases} a_n = a_{0n} e^{-K_{2R} T_1} \\ \theta_n = \left(\frac{1}{4} K_{1I} a_n^2 - K_{2I}\right) T_1 + \theta_{0n} \end{cases} \tag{32}$$

It can be seen that if K_{2R} is positive, the system response will be stable and, if K_{2R} is negative, it will be unstable.

In Table 2, the stability state of the system is listed for different temperatures and velocities for the first two modes. The first mode of the system is unstable for near zero speeds and stable for close to critical speed. Considering the second mode, the system is stable for near zero speeds and the critical velocity. For this mode, the system will be unstable for the speed between fifty and seventy percent of critical velocity. Considering Eqs. 32 and 18, the nonlinear natural frequency is obtained using the amplitude of the time domain response:

Table 2: Stability state

V_0/V_c	first natural frequency	second natural frequency
0	unstable	stable
0.1	unstable	stable
0.3	unstable	stable
0.5	stable	unstable
0.7	stable	unstable
0.9	stable	stable

$$\omega_{nl} = \frac{1-\alpha_T - V_0^2}{\sqrt{1-\alpha_T}} n\pi + \varepsilon\left(\frac{1}{4} K_{1I} a_n^2 - K_{2I}\right) \tag{33}$$

The effect of nonlinear terms on the natural frequencies of the system, considering ideal and non-ideal boundary conditions are shown here in Figures 3–5. In Eq. 33, the nonlinear natural frequencies are related to the parameters K_{1I} , K_{2I}, and the amplitude of vibration. According to the results, if the amplitude increases, the nonlinear natural frequencies of the system increase for ideal and non-ideal boundary conditions. Figures 3 - 5 show that the nonlinear natural frequency for the ideal boundary condition slightly differs from the non-ideal one. The mentioned difference is due to the term K_{2I}.

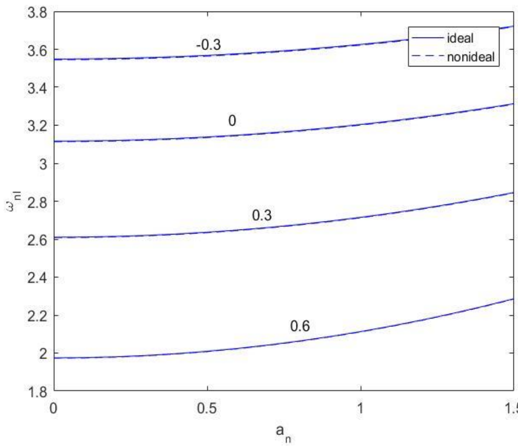

Fig 3: Nonlinear natural frequency versus amplitude in $V_0 = 0.1\ V_c$ at different temperatures α_T

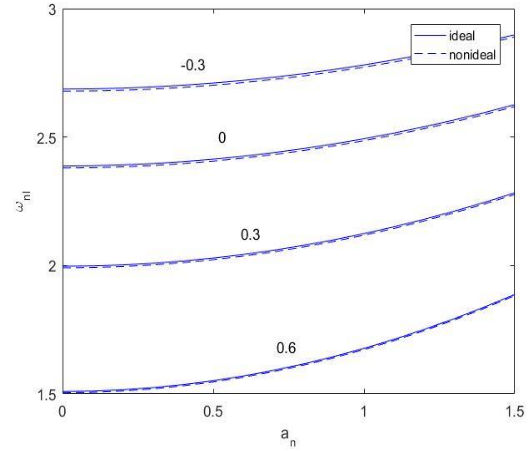

Fig 4: Nonlinear natural frequency versus amplitude in $V_0 = 0.5\ V_c$ at different temperatures α_T

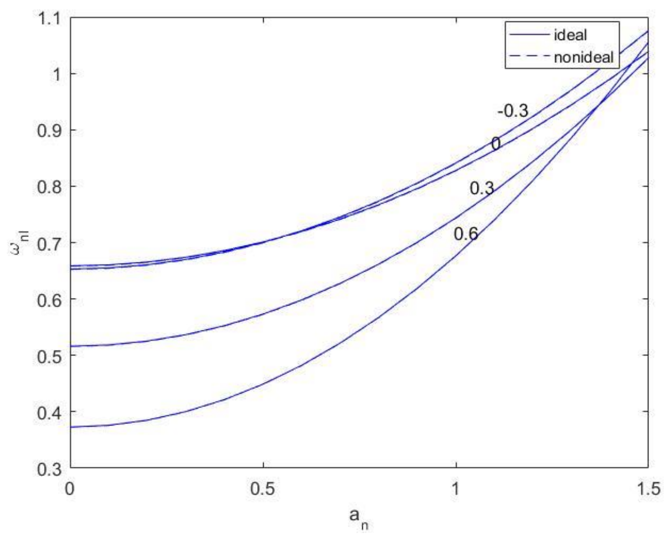

Fig 5 : Nonlinear natural frequency versus amplitude in $V_0 = 0.9\ V_c$ at different temperatures α_T

3.2.2. Ω close to zero

If Ω is close to zero, it can be defined as

$$\Omega = \varepsilon\sigma \tag{34}$$

where σ is the detuning parameter. The solvability conditions for this case are obtained as follows:

$$D_1A+\left(K_3\cos\sigma T_1+K_4\sin\sigma T_1\right)A-K_1A^2\bar{A}+K_2A=0 \tag{35}$$

where K_1 and K_2 are given in Eqs. 29 and 30, and K_3 and K_4 are:

$$K_3=\frac{\int_0^1 V_1\Omega U_0^{'}\bar{U}_0\ dz}{\int_0^1 2\left[i\omega_n\bar{U}_0U_0+V_0\bar{U}_0U_0^{'}\right]\ dz}=K_{3R}+iK_{3I} \tag{36}$$

$$K_4=\frac{\int_0^1 V_1\left(i\omega_nU_0^{'}+2V_0U_0^{''}\right)\bar{U}_0\ dz}{\int_0^1\left[i\omega_n\bar{U}_0U_0+V_0\bar{U}_0U_0^{'}\right]\ dz}=K_{4R}+iK_{4I} \tag{37}$$

By substituting the complex amplitude $A=\frac{1}{2}a_ne^{i\theta_n}$ into Eq. 35, and separating the real and imaginary parts one obtains:

$$D_1a_n=-K_{3R}a_n\cos\left(\sigma T_1\right)-K_{4R}a_n\sin\left(\sigma T_1\right)-K_{2R}a_n \tag{38}$$

$$a_nD\theta_n=K_{3I}a_n\cos\left(\sigma T_1\right)-K_{4I}a_n\sin\left(\sigma T_1\right)+\frac{1}{4}K_{1I}a_n^3-K_{2I}a_n \tag{39}$$

By integrating Eq. 35, the amplitude of vibration becomes:

$$a_n=a_{n0}e^{-\frac{\left(K_{3R}\sin\left(\sigma T_1\right)-K_{4R}\cos\left(\sigma T_1\right)+K_{2R}\sigma T_1\right)}{\sigma}} \tag{40}$$

In Eq. 40, the amplitude is limited due to the fact that $\left|\sin\left(\sigma T_1\right)\right|\le 1$, $\left|\cos\sigma T_1\right|\le 1$. It can be shown that the stability conditions of the system when Ω is close to zero is similar to the case that Ω is far away from zero and $2\omega_n$, and will depend on K_{2R}. So, if it is positive, the system will be stable and if it is negative, the system will be unstable.

3.2.3. Parametric resonance state (Ω close to $2\omega_n$)

In this case, the excitation frequency is considered to be close to twice the natural frequency, therefore

$$\Omega=2\omega_n+\varepsilon\sigma \tag{41}$$

The solvability condition for this case is obtained as follows:

$$D_1A-K_1A^2\bar{A}+K_2A+K_5e^{i\sigma T_1}\bar{A}=0 \tag{42}$$

where K_1 and K_2 are given in Eqs. 29 and 30 and the value of K_5 is:

$$K_5 = V_1 \frac{\int_0^1 [\left(\frac{\Omega}{2} - \omega_n\right)\bar{U}_0^{'}\bar{U}_0 - iV_0\bar{U}_0^{''}\bar{U}_0]dz}{\int_0^1 2\left[i\omega_n \bar{U}_0 U_0 + V_0 \bar{U}_0 U_0^{'}\right] dz} \tag{43}$$

By substituting the complex amplitude $A = \frac{1}{2} a_n e^{i\theta_n}$ into Eq. 39, separating the real and imaginary parts, and considering $\gamma_n = \sigma T_1 - 2\theta_n$, one obtains:

$$Da_n = \left[-K_{2R} - K_{5R}\cos\gamma_n + K_{5I}\sin\gamma_n\right]a_n = G_1 \tag{44}$$

$$a_n D\gamma_n = a_n\sigma + 2a_n\left(K_{5R}\sin\gamma_n + K_{5I}\cos\gamma_n\right) - \frac{1}{2}K_1 a_n^3 + a_n K_{2I} = G_2 \tag{45}$$

For steady state case:

$$Da_n = 0 \tag{46}$$

$$D\gamma_n = 0 \tag{47}$$

For steady state case, there exist trivial and non- trivial solutions. For non-trivial solution, using Eqs. 46 and 47, the relation between σ and $a_n \neq 0$ is obtained as:

$$\sigma_{1.2} = \frac{1}{2}K_{1I}a_n^2 - K_{2I} \pm 2\sqrt{K_{5R}^2 + K_{5I}^2 - K_{2R}^2} \tag{48}$$

To analyze the stability of the non-trivial solution, the Jacobian matrix is evaluated:

$$J = \begin{bmatrix} \frac{\partial G_1}{\partial a_n} & \frac{\partial G_1}{\partial \gamma_n} \\ \frac{\partial G_2}{\partial a_n} & \frac{\partial G_2}{\partial \gamma_n} \end{bmatrix}_{a_n = a_0,\ \gamma_n = \gamma_0} \tag{49}$$

where G_1 and G_2 are given in Eqs. 44 and 45. Eigenvalues of the Jacobean matrix that determine the stability of the solution are:

$$\lambda_{1,2} = -K_{2R} \pm \sqrt{-K_{2R}^2 \pm K_{1I}a_n^2\sqrt{K_{5R}^2 + K_{5I}^2 - K_{2R}^2}} \tag{50}$$

σ_1 and σ_2 are for ideal boundary conditions, and σ_{n1} and σ_{n2} are for non-ideal state. $\sigma = \sigma_1$ and $\sigma = \sigma_{n1}$ are the first bifurcation points. $\sigma = \sigma_2$ and $\sigma = \sigma_{n2}$ are also the second bifurcation points. The region between σ_1 and σ_2, and the region between σ_{n1} and σ_{n2}, respectively, determine the instability of the non- trivial solution in the ideal and non-ideal states.

Figures 6-9 exhibit bifurcation phenomenon. In these graphs, the instability regions represent non- trivial solution of the system in the parametric resonance state. In Figure 6, the system with ideal boundary conditions is considered at speeds far away from critical speed. In this case, the curves related to σ_{n1} and σ_{n2} coincide. Therefore, the trivial solution will be valid.

Figure 7 shows the system with ideal boundary conditions at a speed far from the critical velocity. In this case, σ_1 and σ_2 are different. The system will have a non-trivial solution in the region between the two curves that correspond to σ_1 and σ_2. This region decreases with increasing temperature.

In Figure 8, the system with non-ideal boundary conditions is considered at a speed close to the critical velocity. Under this condition, there is a difference between the results related to σ_{n1} and σ_{n2} which refers to the validity of the non-trivial solution in this region. In Figure 9, the system with ideal boundary conditions is considered at a speed close to the critical velocity. In this case, there is also a difference between the curves of σ_1 and σ_2. Similar to the previous case, this difference points the validity of a non-trivial solution in this region. By comparing the results of the ideal and non-ideal boundary conditions, it can be inferred that in the ideal state at all speeds, there is a difference between σ_1 and σ_2 curves. The region between σ_{n1} and σ_{n2} is asymmetric in the system with non-ideal boundary condition whereas region between σ_1 and σ_2 is symmetric in the system with ideal boundary conditions.

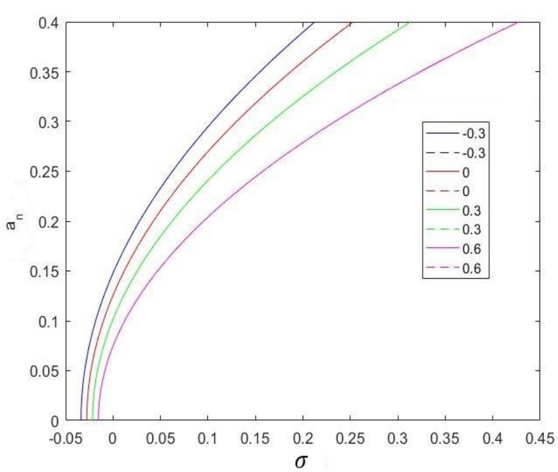

Fig 6: Amplitude versus σ for non-ideal state in $V_0 = 0.1\ V_c$ at different temperatures α_T, σ_{n1} (−), σ_{n2} (--)

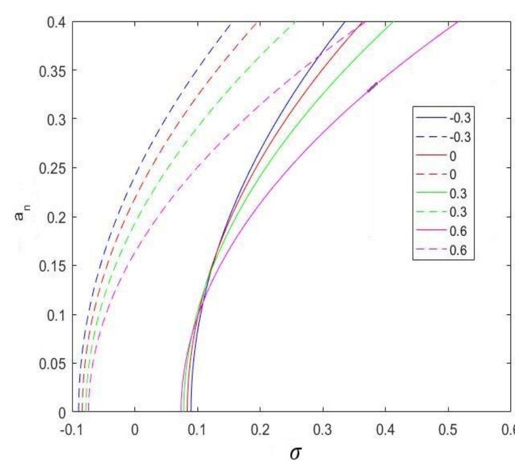

Fig 7: Amplitude versus σ for ideal state in $V_0 = 0.1\ V_c$ at different temperatures α_T, σ_1 (−), σ_2 (--)

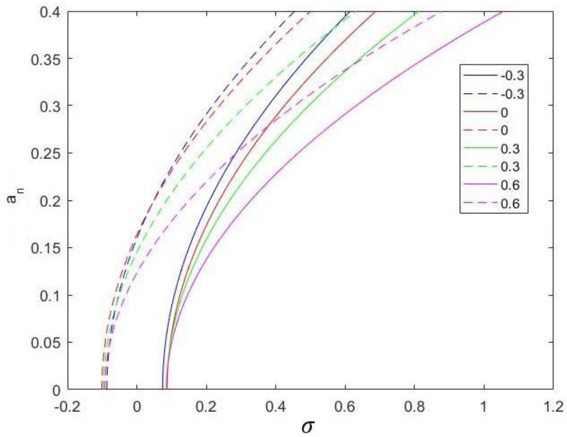

Fig 8: Amplitude versus σ for non-ideal state in $V_0 = 0.9\, V_c$ at different temperatures α_T, σ_{n1} (−), σ_{n2} (--)

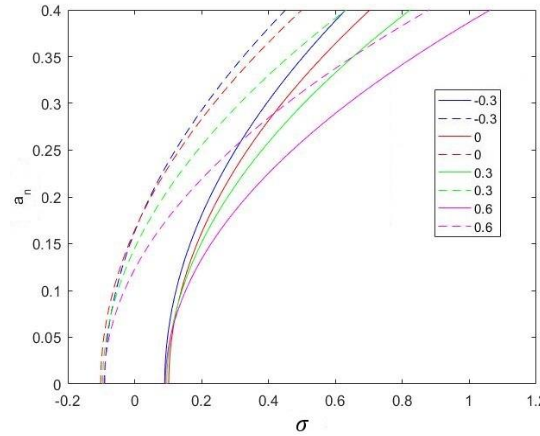

Fig 9: Amplitude versus σ for ideal state in $V_0 = 0.9\, V_c$ at different temperatures α_T, σ_1 (−), σ_2 (--)

The stable region for trivial solution is determined by performing stability analysis. For trivial solution, the amplitude is considered as zero, therefore:

$$A = \frac{1}{2}(p+iq)e^{i\frac{\sigma}{2}T_1},\ a_n = p+iq = 0:\ \text{trivial solution} \tag{51}$$

By substituting Eq. 51 into Eq. 42, and separating the real and imaginary parts, one obtains:

$$D_1 p = -\left(K_{2R}+K_{5R}\right)p + \left(\frac{\sigma}{2} - K_{5I} + K_{2I}\right)q - \frac{1}{4}K_{1I}q\left(p^2+q^2\right) = F_1 \tag{52}$$

$$D_1 q = \left(K_{5R}-K_{2R}\right)q - \left(\frac{\sigma}{2} + K_{5I} + K_{2I}\right)p + \frac{1}{4}K_{1I}p\left(p^2+q^2\right) = F_2 \tag{53}$$

The Jacobian matrix for this state is obtained as follows:

$$\begin{bmatrix} \frac{\partial F_1}{\partial p} & \frac{\partial F_1}{\partial q} \\ \frac{\partial F_2}{\partial p} & \frac{\partial F_2}{\partial q} \end{bmatrix} = \begin{bmatrix} -(K_{5R}+K_{2R}) & \frac{\sigma}{2} - K_{5I} + K_{2I} \\ -\left(\frac{\sigma}{2} + K_{5I} + K_{2I}\right) & K_{5R} - K_{2R} \end{bmatrix}_{p=q=0} \tag{54}$$

Eigenvalues of the Jacobian matrix are:

$$\lambda_{1,2} = K_{2R} \pm \sqrt{K_{5R}^2 + K_{5I}^2 - (\frac{\sigma}{2} + K_{2I})^2} \tag{55}$$

In this stage, $\lambda = 0$ is taken for determining the stability boundaries. The stability region of the system for trivial solution is determined as:

$$\begin{cases} \sigma > -2K_{2I} + 2\sqrt{K_{5R}^2 + K_{5I}^2 - K_{2R}^2} \\ \sigma < -2K_{2I} - 2\sqrt{K_{5R}^2 + K_{5I}^2 - K_{2R}^2} \end{cases} \tag{56}$$

If σ is outside the region provided by Eq. 56, the trivial solution will be unstable. Thus, the region of instability is:

$$-2K_{2I} - 2\sqrt{K_{5R}^2 + K_{5I}^2 - K_{2R}^2} < \sigma < -2K_{2I} + 2\sqrt{K_{5R}^2 + K_{5I}^2 - K_{2R}^2} \tag{57}$$

This region is important because when the trivial solution is unstable, a non- trivial solution should be considered. Finally, using Eq. 41, the stability region is determined as:

$$\Omega = 2\omega_n \pm \varepsilon\left(2K_{2I} \pm 2\sqrt{K_{5R}^2 + K_{5I}^2 - K_{2R}^2}\right) \tag{58}$$

4. Conclusion

In this study, nonlinear vibration of an axially moving string in thermal environment has been investigated. The axial velocity is assumed to be summation of a constant value and a small-amplitude harmonic function. It is also assumed that there is a thermal balance between the system and the environment. Following modeling of the system which is suitable for many real systems, the equation of motion is derived using the Hamilton's principle. The dimensionless form of the equation of motion is then derived and then solved by the MTS method.

From the solution of linear problem, natural frequencies, mode shape functions and critical velocity for various values of effective parameters such as temperature and velocity of the system are obtained and tabulated. According to the results, natural frequencies and critical speeds decrease if the temperature increases. The justification is that the material becomes softer with increasing temperature. Also, the natural frequencies decrease if the axial velocity increases. The effect of nonlinear terms appears in higher order of solution. The solution of problem is found in three cases; Ω far from zero and $2\omega_n$, Ω close to zero, and Ω close to $2\omega_n$ (the parametric resonance case).

If Ω is far away from zero and $2\omega_n$, instability occurs when K_{2R} is negative. For example, around the first mode, if the axial velocity is less than fifty percent of the critical velocity related to any temperature, the system becomes unstable and the amplitude increases with time. This description is valid for the non-ideal boundary conditions; because for ideal boundary conditions,

K_{2R} is zero and the amplitude remains constant over time. For this case, the nonlinear natural frequencies for the ideal and non-ideal boundary conditions are dependent on the amplitude of vibration. The difference between the ideal and non-ideal states decreases with increasing temperature. If speeds are close to zero and the critical velocity, the difference between nonlinear natural frequencies in ideal and non-ideal cases become very small and this difference is more when the speed is close to fifty percent of the critical velocity. It is found that, when Ω is close to zero, at some speeds, the system becomes unstable or stable for different temperatures. In this case, stability state is the same as when Ω is far from zero and $2\omega_n$.

For the parametric resonance case, the region where the non-trivial solution exists is obtained analytically. The bifurcation diagrams for ideal and non-ideal boundary conditions are presented under the influence of temperature at various speeds. The results show that for non-ideal boundary conditions, at any temperature near critical velocities a bifurcation phenomenon occurs, whereas it does not occur at lower velocities. On the other hand, with ideal boundary conditions, bifurcation phenomenon occurs always.

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