Journal of Theoretical and Applied Vibration and Acoustics 5(1) 11-20 (2019)

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Journal of Theoretical and Applied Vibration and Acoustics

journal homepage: http://tava.isav.ir

System identification of a beam with frictional contact

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ARTICLE INFO

Article history:
Received 27 May 2019

Received in revised form
22 June 2019

Accepted 25 June 2019

Available online 27 June 2019

Keywords:
Nonlinear boundary condition,

System identification,

Multiple scales method.

ABSTRACT

The nonlinear system becomes an area with numerous investigations over the past decades. The conventional modal analysis could not be applied on nonlinear continuous system which makes it impossible to construct the reduced order models and obtain system response based on limited number of measurement points. Nonlinear normal modes provide a useful tool for extending modal analysis to nonlinear systems but the extraction of nonlinear normal modes is cumbersome. In this research by neglecting the damping effect on response shape, the nonlinear joint in frictional beam is replaced by 3 order nonlinear spring. The equation of motion is solved using the method of multiple scales up to third super harmony which are neglected in most system identification practices. It is shown in an experimental test set-up, the general form of solution can regenerate observed response of any point and more important similar to linear system, the single mode would be sufficient in analysis when excitation frequency is close to resonant frequency.

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1. Introduction

Supports have an important role in system dynamic behavior and must be considered carefully when it comes to constructing analytical or numerical models[1-3]. In the most models, supports are supposed to behave linearly. Linearity is an idealization and supports can cause nonlinearity in the response and dissipation of the energy due to nonlinear mechanisms developing at their contact interfaces. Neglecting the nonlinear characteristics of the boundary conditions is the main source of discrepancies between experimental observations and the results obtained by using analytical or numerical models.

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http://dx.doi.org/10.22064/tava.2019.113938.1146

Any attempt to apply traditional linear analysis to nonlinear systems results, at the best, in a suboptimal design. Thus, there is a need for efficient, analytically rigorous, and broadly applicable analysis techniques for nonlinear structural dynamics. In this context, nonlinear normal modes offer mathematical tool for interpreting a wide class of nonlinear dynamical phenomena, however, the heavy mathematical calculations restrict its application in most cases. A comprehensive literature survey about nonlinear normal modes has been presented by Vakakis [4]. The review of the recent contributions in this field can be found in [5, 6].

In experimental approach, Ahmadian *et al*.[7] employed the equivalent linearization method to identify the single nonlinear normal mode. The method assumes the nonlinear normal mode is a combination of the base linear system normal modes. In this method, by using the measured responses, the contribution of each linear mode shape in nonlinear normal mode is determined. The higher harmonics are neglected which could have significant contribution in the response.

This paper considers the nonlinear behavior of a beam with frictional contact support. The structure nonlinearity is local. The identification of contact support requires the joint displacement which are impossible to measure directly. The joint response must be evaluated as the function of measurement points on the beams. Jalali *et al*. [8] investigated this set-up by the force state mapping method. Turner [9] studied the multiple scale method application to solve the beam with Hertzian contact which is capable of considering the higher harmonies.

The remaining of this paper is organized as follows. In the next section, the appropriate boundary value problem is described. The method of multiple scales is used to analyze this problem and the general form of solution is evaluated. The primary results from this analysis include the amplitude relation for the various flexural modes. Next, an experimental case study is considered and unknown point response is obtained based on two measurement points. Conclusions are drawn and references are presented.

2. Theory

This paper considers the dynamic behavior of a slender beam clamped on one end and frictionally supported at the other end as shown in Fig. 1.The dynamic response of beam is modeled using the Euler–Bernoulli beam theory. The geometrical and material properties of the beam are mass density ρ, modulus of elasticity *E*, cross sectional moment of inertia *I*, cross section area *A*, and length *L*. The frictional contact boundary condition is provided using a rod attached to the beam end. The structure is excited using a concentrated moment *M(t)* in the presence of the constant force as a contact preload to restrict the lateral movement of the beam at frictional support. The equation governing dynamic response of the test structure can be expressed as:

$$\frac{EI}{\rho A}\, y''''(x,t) + \ddot{y}\,(x,t) = 0. \tag{1}$$

The frictional support is modeled by a 3-order nonlinear torsional spring $(k_{1\theta}\theta + k_{2\theta}\theta^2 + k_{3\theta}\theta^3)$ and linear damper C. The Eq. (1) is subjected to the following boundary and compatibility conditions:

$$y(0,t) = y'(0,t) = y(L,t) = 0, \tag{2}$$

$$y''(L,t) = k_1 y'(L,t) - \varepsilon\, k_2 \big(y'(L,t)\big)^2 - \varepsilon^2\, k_3 (y'(L,t))^3 + \varepsilon^2\, c(\dot{y})'(L,t) + \varepsilon^2\, \frac{M(t)}{EI}, \tag{3}$$

$$k_1 = \frac{k_{1\theta}}{EI} \quad k_2 = \frac{k_{2\theta}}{EI} \quad k_3 = \frac{k_{3\theta}}{EI} \quad c = \frac{C}{EI}. \tag{4}$$

where ε is a small dimensionless variable applied to order the different scales of the equation. The damper force, applied moment and the highest order of nonlinearity should have the same order to achieve a uniformly valid solution [10].

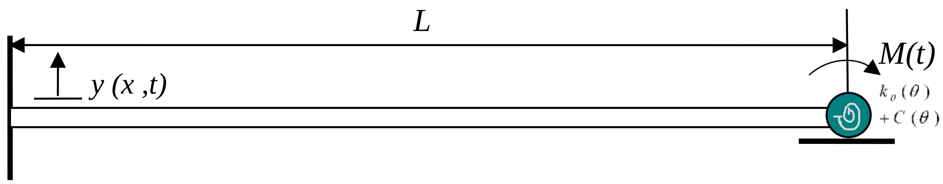

Fig. 1: A schematic view of the beam.

The nonlinear boundary value problem (1) could be solved using the method of multiple scales [10]. We look for solutions of the form:

$$(x;T_0,T_1,T_2) = y_0(x;T_0,T_1,T_2) + \varepsilon\, y_1(x;T_0,T_1,T_2) + \varepsilon^2\, y_2(x;T_0,T_1,T_2). \tag{5}$$

where $T_n = \varepsilon^n t$. By substituting Eq. (5) in Eq. (1) and gathering the same order terms, three equations would be achieved. As a result the zero-order problem is:

$$\frac{EI}{\rho A}\, y_0''''(x,t) + D_0^2 y_0 = 0, \tag{6}$$

$$y_0(0,t) = y_0'(0,t) = y_0(L,t) = 0, \tag{7}$$

$$y_0''(L,t) = k_1 y_0'(L,t). \tag{8}$$

The first order equation describes as:

$$\frac{EI}{\rho A}\, y_1''''(x,t) + D_0^2 y_1 = -2\, D_0 D_1 y_0, \tag{9}$$

$$y_1(0,t) = y_1'(0,t) = y_1(L,t) = 0, \tag{10}$$

$$y_1''(L,t) = k_1 y_1'(L,t) - k_2 \big(y_0'(L,t)\big)^2. \tag{11}$$

And the second-order equation expresses as:

$$\frac{EI}{\rho A}\, y_2''''(x,t) + D_0^2 y_2 = -2\, D_0 D_1 y_1 - (D_1^2 + 2 D_0 D_2) y_0 + M cos(\Omega t), \tag{12}$$

$$y_2(0,t) = y_2'(0,t) = y_2(L,t) = 0, \tag{13}$$

$$y_2''(L,t) = k_1 y_2'(L,t) - k_2 y_0'(L,t) y_1'(L,t) - k_3 (y_0'(L,t))^3 + c\, D_0 y_0'. \tag{14}$$

3. Response solution

3.1.The zero-order equation solution

The solution to Eq. (6) of ε^0 is as follows:

$$y_0(x;T_0,T_1) = A(T_1,T_2)e^{i\omega T_0}\, w(x) + cc, \tag{15}$$

where ω is natural frequency of based linear system, A is the amplitude as function of two slow time scales T_1 and T_2 and cc denotes the complex conjugate of the previous terms. Substituting Eq. (15) in Eq. (6) leads to:

$$w'''' - \lambda^4 w = 0. \tag{16}$$

The boundary conditions have also been rewritten as:

$$w(0) = w'(0) = w(L) = 0, \tag{17}$$

$$w''(L) = k_1 w'(L). \tag{18}$$

The solution to set of Eqs. (16-8) is:

$$k_1(1 - \cos\lambda L \cosh\lambda L) - \lambda(\cosh\lambda L \sin\lambda L - \cos\lambda L \sinh\lambda L) = 0, \tag{19}$$

where $\lambda^4 = \omega^2 \rho A / EI$. The parameter λ could be evaluated by the characteristic equation obtained from the linear problem. This equation is a function of the linear spring stiffness k_1 as follows:

$$k_1(1 - \cos\lambda L \cosh\lambda L) - \lambda(\cosh\lambda L \sin\lambda L - \cos\lambda L \sinh\lambda L) = 0. \tag{20}$$

3.2.The first-order equation solution

The solution of y_0 is necessary to solve the equations of the order ε^1. Substituting Eq. (15) in Eqs. (9-11) leads to:

$$\frac{EI}{\rho A}\, y_1''''(x,t) + D_0^2 y_1 = -2\, i\omega A e^{i\omega T_0} w(x), \tag{21}$$

$$y_1(0,t) = y_1'(0,t) = y_1(L,t) = 0, \tag{22}$$

$$y_1''(L,t) = k_1 y_1'(L,t) - k_2\left(A^2 e^{2i\omega T_0} + 2A\overline{A} + \overline{A}^2 e^{-2i\omega T_0}\right) w'(L). \tag{23}$$

We seek solutions of the form:

$$y_1(x;T_0,T_1,T_2) = q_1(x;T_1,T_2)e^{i\omega T_0}. \tag{24}$$

Substituting y_1 at Eq. (21), the equation is rewritten:

$$q_1'''' - \lambda^2 q_1 = -2i\omega D_1 A w(x), \tag{25}$$

$$q_1(0;T_1,T_2) = q'_1(0;T_1,T_2) = q_1(L;T_1,T_2) = 0, \tag{26}$$

$$q''_1(L;T_1,T_2) = k_1 q'_1(L;T_1,T_2). \tag{27}$$

The inhomogeneous Eq. (25) only has an answer when a solvability condition is established. Here, the solvability condition requires that D1A be equal to zero. This means that the parameter A is only dependent on T2. The second part of the solution y1 for the set of Eqs. (21-23) is defined as follows:

$$y_1 = q_2(x)e^{2i\omega T_0}. \tag{28}$$

Substituting Eq. (28) in Eqs. (21-23), the forth order ODE and corresponding boundary conditions would be achieved and q2 can be expressed as:

$$q_2(x) = \frac{k_2A^2}{2}\ [\ C_1\,(sin\,\lambda_2 x - sinh\,\lambda_2 x) +\ C_2\,(cos\,\lambda_2 x - cosh\,\lambda_2 x)\ ] = \frac{k_2A^2}{2}\ [\,Q(x,\lambda_2)\,], \tag{29}$$

where

$$\lambda_2 = \lambda\sqrt{2}, \tag{30}$$

$$C_1 = \frac{-(cos\,\lambda_2 x - cosh\,\lambda_2 x),}{k_1\lambda_2(1 - cos\,\lambda_2 L\,cosh\,\lambda_2 L\,) - \lambda_2^2\,(cosh\,\lambda_2 L\,sin\,\lambda_2 L - cos\,\lambda_2 L\,sinh\,\lambda_2 L),} \tag{31}$$

$$C_2 = \frac{-(sin\,\lambda_2 x - sinh\,\lambda_2 x),}{k_1\lambda_2(1 - cos\,\lambda_2 L\,cosh\,\lambda_2 L\,) - \lambda_2^2\,(cosh\,\lambda_2 L\,sin\,\lambda_2 L - cos\,\lambda_2 L\,sinh\,\lambda_2 L).} \tag{32}$$

Q is a function of x and $\lambda_2^4 = 4\lambda^4$; also, the stiffness coefficient of the second-order spring k_2 is seen in q_2. C_1 and C_2 are constant coefficients and could be obtained by applying boundary conditions at this stage.

3.3.The second-order equation solution

As before, in this section, the solution y_0 and y_1 are necessary to solve the equation of the order ε^2. Substituting Eqs. (24,28) in Eqs. (12-14) leads to:

$$\frac{EI}{\rho A}\,y_2''''(x,t) + D_0^2 y_2 = -2i\omega D_2 A e^{i\omega T_0} w(x), \tag{33}$$

$$y_2(0,t) = y_2'(0,t) = y_2(L,t) = 0, \tag{34}$$

$$\begin{aligned} y_2''(L,t) = k_1 y_2'(L,t) - \left(\frac{k_2^2 A^2\overline{A}}{2} Q'(L,\lambda_2) - 3k_3A^2\overline{A} + ci\omega A\right) e^{i\omega T_0}\,w'(L) \\ + \left(\frac{-k_2^2A^3}{2} Q'(L,\lambda_2) - k_3A^3\right) e^{3i\omega T_0}\,w'(L) + \frac{M(t)}{EI} cos(\Omega t). \end{aligned} \tag{35}$$

First, we seek the solution as a form:

$$y_2(x;T_0,T_2) = q_3(x;T_2)e^{i\omega T_0.} \tag{36}$$

Substituting Eq. (36) in Eq. (33-35) leads to:

$$q_3'''' - \lambda^4 q_3 = -2i\omega D_2 A w(x), \tag{37}$$

$$q_3(0;T_2) = q'_3(0;T_2) = q_3(L;T_2) = 0, \tag{38}$$

$$q_3''(L;T_2) = k_1 q_3'(L;T_2) + \frac{M(t)}{EI}cos(\Omega t) - \left(\frac{k_2^2 A^2 \overline{A}}{2} Q'(L,\lambda_2) - 3k_3 A^2 \overline{A} + ci\omega A\right) e^{i\omega T_0}\, w'(L). \tag{39}$$

The inhomogeneous Eq. (37) has the solution only when a solvability condition is established. This condition is expressed as:

$$-\left(\frac{k_2^2 A^2 \overline{A}}{2} Q'(L,\lambda_2) - 3k_3 A^2 \overline{A} + ci\omega A\right) w'(L) = i\omega\, D_2 A\, G(\lambda, k_1) - \frac{M}{2\lambda^4 EI} e^{i\Omega T_0 - i\omega T_{0.}} \tag{40}$$

where

$$\begin{aligned} &G(\lambda,k_1)\\ &= k_1\left(\frac{(cos\,\lambda L - cosh\,\lambda L) + \dfrac{(sin\,\lambda L\; - sinh\,\lambda L)\,(sin\,\lambda L\; + sinh\,\lambda L)}{(cos\,\lambda L - cosh\,\lambda L)}}{2\lambda^3}\right)\\ &- \frac{L\left[(\lambda\, sin\,\lambda + \lambda\, sinh\,\lambda L) - \left(\dfrac{\left((sin\,\lambda L\; - sinh\,\lambda L)(\lambda\; cos\,\lambda L + \lambda\; cosh\,\lambda L)\right)}{cos\,\lambda L\; - cosh\,\lambda L}\right)\right]}{2\lambda^3}\\ &+ \left(\frac{\dfrac{\lambda\, sinh\,\lambda L + \lambda\, sinh\,\lambda L}{(cos\,\lambda L\; - cosh\,\lambda L)} - \dfrac{-\left((sin\,\lambda L\; - sinh\,\lambda L)(\lambda\, cos\,\lambda L\; +\; \lambda\, cosh\,\lambda L)\right)}{(cos\,\lambda L\; - cosh\,\lambda L)}}{\lambda^3}\right)\\ &+ \frac{L\left(\lambda^2\, cos\,\lambda L + \;\lambda^2\, cosh\,\lambda L + \dfrac{\left((sin\,\lambda L\; - sinh\,\lambda L)(\lambda^2\, sin\,\lambda L\; -\; \lambda^2\, sinh\,\lambda L)\right)}{(cos\,\lambda L\; - cosh\,\lambda L)}\right)}{2\lambda^3}. \end{aligned} \tag{41}$$

Applying the boundary condition in Eq. (37), q_3 can be expressed as:

$$q_3 = \left(\frac{k_2^2 A^2 \overline{A}}{2} Q'(L,\lambda_2) - 3k_3 A^2 \overline{A} + ci\omega A\right) N(\lambda, x) - \frac{1}{2\lambda^4 EI}\left(M e^{i\Omega T_0 - I\omega T_0} + cc\right), \tag{42}$$

where

$$N(\lambda,x) = -\frac{Q(\lambda,x)}{2} - \frac{1}{G2\lambda^3}\left[x(cos\,\lambda x - cosh\,\lambda x) - x(sin\,\lambda x + sinh\,\lambda x)\left(\frac{sin\,\lambda L - sinh\,\lambda L}{cosh\,\lambda L - cos\,\lambda L}\right)\right], \tag{43}$$

In the Following, the particular solution of y2 should be obtained:

$$y_2 = q_4(x)\, e^{3i\omega T_0}, \tag{44}$$

$$q_4 = \frac{-A^3}{2} Q(x,\lambda_3)(-k_3 + k_2^2 Q(L,\lambda)), \tag{45}$$

$$\lambda_3 = \lambda\sqrt{3}. \tag{46}$$

At the end, the solution of y_2 of second order ε, obtained from the sum of the Eq. (44) and Eq. (36):

$$y_2 = q_3 e^{i\omega T_0} + q_4 e^{3i\omega T_0}. \tag{47}$$

3.4. *The complete form of solution*

Having the solution of y_0, y_1 and y_2 enables us to form the complete form of solution:

$$\begin{aligned} Y(x;T_0,T_2) = &\left(w(x)A + \left(\frac{k_2^2 A^2 \overline{A}}{2} Q'(L,\lambda_2) - 3k_3 A^2 \overline{A}\right) N(\lambda,x) + ic\omega A - \frac{M}{2\lambda^4 EI} e^{i\Omega T_0}\right) e^{i\omega T_0} \\ &+ \left(\frac{k_2 A^2}{2} Q(x,\lambda_2)\right) e^{2i\omega T_0} + \left(\frac{-A^3}{2} Q(x,\lambda_3)(-k_3 + k_2^2 Q'(L,\lambda_2)\right) e^{3i\omega T_0}. \end{aligned} \tag{48}$$

If the amplitude A, rewrites in polar form:

$$A = \frac{1}{2} a e^{i\beta}. \tag{49}$$

where a and β are function of T_2. Substituting Eq. (49) in Eq. (48) lead to:

$$\begin{aligned} (x;T_0,T_2) = &\left(w(x)a + \left(\frac{k_2^2}{2} Q'(L,\lambda_2) - 3k_3\right) N(\lambda,x) \frac{1}{4} a^3 + ic\omega a + \frac{M}{2\lambda^4 EI} e^{i\Omega T_0}\right) cos(\omega T_0 + \beta) \\ &+ \left(\frac{k_2}{2} Q(x,\lambda_2) a^2\right) cos(2\omega T_0 + 2\beta) \\ &+ \left(Q(x,\lambda_3)(k_3 - k_2^2 Q'(L,\lambda_2) \frac{1}{8} a^3\right) cos(3\omega T_0 + 3\beta). \end{aligned} \tag{50}$$

The Eq. (50) expresses the amplitude as a nonlinear function of stiffness of linear and nonlinear torsional springs and damping coefficient. The response included terms of a modification to the harmonic linear mode with a second and third harmonic components. One may interprets this relation as nonlinear normal mode. In Eq. (50), the functions G, N and Q represent the dependence of the nonlinear response on the different harmonics. The first harmonic is proportional to the function $N(\lambda, x)$. The second harmonic include squared a and is proportional to the function $Q(x, \lambda_2)$ and finally the third harmonic is proportional to the function $Q(x, \lambda_3)$.

4. Experimental test

The set-up used in this section is a beam that is similar to the analytical model. The excitation force is single harmony and its frequency is close to first natural frequency of linear based system. The nonlinear behavior of the beam can be investigated using nonlinear Eq. (50).

A uniform beam with a length of *600 mm*, a width of *40 mm* and a thickness of *5 mm* is clamped on one side and has frictional contact on the other end as shown in Fig. 2. The rod has a radius of *5 mm* and a length equal to the width of the beam. The force applied by a *B&K 4200* minishaker attached at a distance *S=570 mm* from the clamped end and a *B&K 8200* force transducer was attached between the stinger and the beam to measure the input force. The distance of excitation force to the beam end is only *3 cm*, so it could be supposed as a bending moment applied on beam end similar to analytical model. The response was measured using three *AJB 120* accelerometers mounted on the beam at locations $x_1 = 100$, $x_2 = 300$ and $x_3 = 570$ *mm* from the clamped end.

Fig. 2: Test set-up.

The unknown parameters in Eq. (50) are as follows: *a* amplitude, k_1 linear spring stiffness, k_2 and k_3 nonlinear second and third order spring stiffness respectively and damping coefficient *c*. These parameters should be identified to form the response in arbitrary location of beam.

The excitation frequency was considered *54.2 Hz* to be close to first resonance frequency. The excitation level adjusted to have *5g* response at the direct accelerometer. Three accelerometers and force transducer output is shown is Fig. 3.

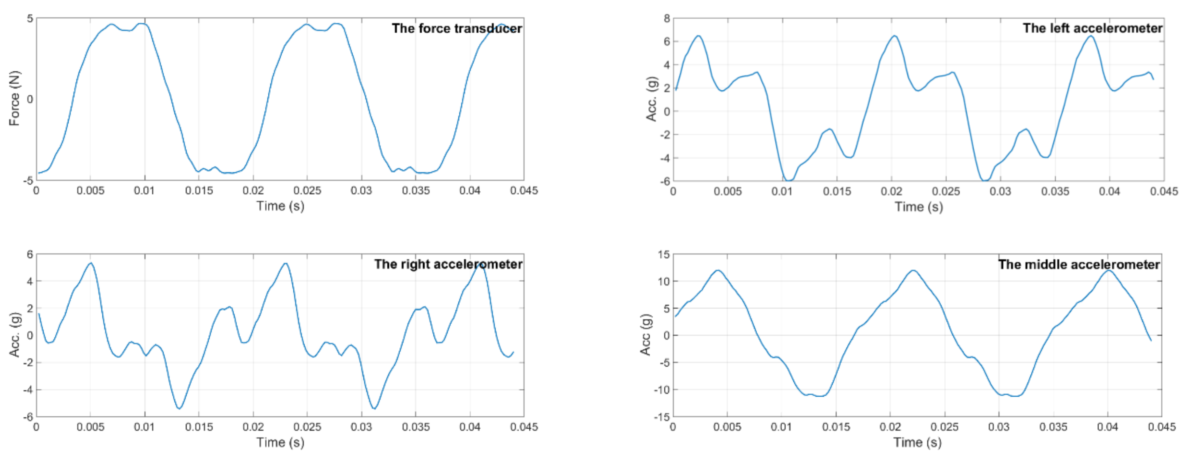

Fig. 3: Time domain response of accelerometers and force transducer.

The unknown parameters are identified by *Gaoptimset* function from *MATLAB* genetic algorithm toolbox 2016. The optimization input is force transducer record and choose the parameters to have minimum error between output of Eq. (50) and two accelerometers on the left. The identified parameters are shown in Table 1.

Table1: The identified parameters.

c	K_3	K_2	K_1	**a**
71 (N.s/m)	-329(N/m)	51(N/m)	1682 (N/m)	**2.6m**

The identified quadratic term in stiffness is small compared to cubic term because friction is a symmetric function. Parameter values are used to estimate the response of the third accelerometer. Fig. 4 compares the results of the test data with the simulation results. The solid line corresponds to the simulation results by Eq. (50) and the dotted line to the results of the test data. It is clear that there is good agreement between the results which imply that the parameters have been successfully identified. Given that the analytical model expansion limited the third harmony, the simulation predicted these harmonies with good accuracy and the errors between responses arising from higher harmonics.

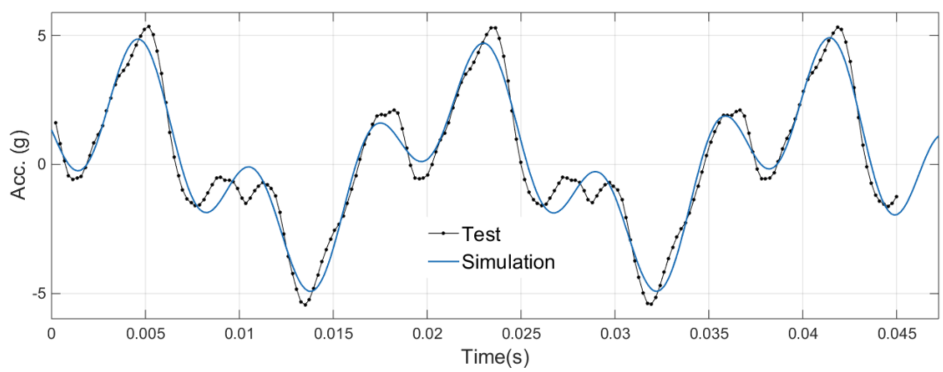

Fig. 4: the comparison of test and prediction for the right accelerometer.

It is worth to mention for simplification, a linear damper was considered. A nonlinear damper leads to the cumbersome relations but as an outcome, the simulation would be able to predict the response at different excitation levels.

5. Conclusion

The dynamic characteristics of a beam with frictional contact support are investigated. The frictional support is replaced with nonlinear torsional spring containing quadratic and cubic terms. This simplification enables us to solve the dynamic equation of beam using the method of multiple scales. The mathematical model of the structure is employed in experimental set-up to predict the displacement of one point as a function of other points. The method imports terms of a modification to the harmonic linear mode which are neglected in most identification methods. The obtained analytical model is capable of regenerating the responses and the errors are the effect of higher harmonies which are omitted.

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