

# Design and Simulation of Acoustic Metamaterial Luneburg Lenses for Predetermined Focal Points

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## Abstract

This paper presents the design and simulation of acoustic metamaterial lenses that focus elastic waves at predetermined focal points. The modified Luneburg refractive index profile is used in the design process to define the focal point locations, which is a capability not previously explored in elastic wave research. This new approach is important because it enables more precise spatial control of waves, resulting in enhanced resolution for elastic wave focusing applications. Three lenses, each targeting specific focal points, are designed by proposing hexagonal unit cells containing blind holes with varying diameters. Dispersion curves are calculated by finite element simulations to determine wave properties of unit cells, including refractive indices. These unit cells provide a wide range of refractive indices (1.0314-1.4959) at the design frequency of 50 kHz which is suitable for constructing Luneburg lenses. Unit cells are then arranged according to the discretized refractive index profiles to form the lenses. Numerical simulations validate effective wave focusing at the intended focal points ( $F=R$ ,  $1.5R$ ,  $2R$ ) with three lenses. The highest amplification of waves and narrowest focal zone is for the lens with  $F=R$ . As focal point shifts toward  $2R$ , wave distribution becomes scattered along the focal axis. Decay length analysis of  $F=1.5R$  and  $2R$  lenses indicates their suitability for long distribution of high-velocity regions. Frequency-dependent simulations across 46–52 kHz reveal all lenses maintain efficient focusing between 49–51 kHz. At more distant off-design frequencies, amplifications result from refractive index shifts that misalign the focal point.

**Keywords:** Acoustic Metamaterials; Luneburg lens; Predetermined Focal Points; Elastic Waves.

## 1. Introduction

In recent years, there has been a growing interest in the field of acoustic metamaterials (AMs). AMs are artificial materials that can control sound waves through innovative mechanisms like local resonances or geometrical arrangements, often achieving unnatural effects like negative refraction. Acoustic metamaterials have potential applications in various industrial and medical fields[1–5] as acoustic absorbers[6], barriers[7], lenses[8–10] and cloaks[11]. Acoustic lenses are mainly devices that are specifically designed to focus sound waves in one or multiple focal points. The unique properties of lenses are useful for elastic wave applications like sensing and damage detection in solids[12], non-destructive testing[13], and energy harvesting[14]. Current study demonstrates the potential of these lenses in precisely controlling acoustic/elastic wave propagation.

The literature surrounding acoustic metamaterial lenses highlights various methods of wave control which lead to wave focusing. In a related study at 2022, Zhao et al. [15] explored the Rays Inserting Method (RIM), to design acoustic lenses for flexural wave manipulation by varying the thickness of thin plates to achieve a desired refractive index profile. Additionally, Yang et al. [16] designed a negative refraction index lens by proposing a structure of rotating scatterers on a plate containing periodic perforated three-fan holes. Scatterers with different rotation angles exhibit variations of the dispersion curve, which is related to a specific negative refraction index. Their designed lens had a fixed focal point in the frequency range of 16 to 19kHz and was

analysed for energy harvesting. In a paper at 2024, Quadrelli et al. [17] reported the development of a metamaterial with both negative dynamic mass and stiffness which was capable of negative refraction of elastic flexural waves, eventually resulting in a focusing lens. The double negative behaviour was obtained by both mechanical resonators and piezoelectric patches with inductive resonant shunts. Zhao et al. [18] in their 2023 study, introduced a structural Luneburg lens for broadband ultralong subwavelength focusing, utilizing two concentric circular regions with varying thickness in a thin plate to achieve double foci and control energy flow between the two focal spots. Their results showed the successful achievement of a large full length at half maximum (*FLHM*) up to  $15\lambda$  at the focusing region, while the full width at half maximum (*FWHM*) is limited to a subwavelength scale. It is worth mentioning that their primary focus was on achieving a long focal region rather than precise control over the exact locations of the focal points, the latter one being the main target of the present study. Tol et al. [19] designed a gradient index lens for elastic wave energy harvesting, using an array of blind holes with varying diameters in an aluminium plate, though their lens lacked omnidirectionality and used square unit cells. Building on this, Tol et al. [20] developed a phononic crystal Luneburg lens with omnidirectional elastic wave focusing, though without control over the focal point.

This study presents a lens design method based on a structure comprising hexagonal unit cells with blind holes, which offers greater structural integrity compared to previous fully perforated and variable thickness structures. Adjusting the diameter of blind holes of this type of unit cell provides the desired acoustic properties of refractive index for the lens design. This offers a wide range of refractive indices at the design frequency of  $50\text{ kHz}$ . The proposed design method of this paper provides the capability to achieve arbitrary positioned focal points of elastic waves which is crucial for precise elastic wave control in real-world applications of the lens. This is achieved with different variants of the proposed structure to approximate modified Luneburg refractive index profile. For this reason, three distinct lenses are designed. Their performance is validated through finite element simulations in COMSOL. In addition to evaluating the performance of the lenses at the design frequency, a frequency-dependent study is conducted to observe how changes in frequency affect wave concentration behaviour. In the subsequent sections, section 2 comprehensively explains the procedure for lens design, section 3 covers the finite element simulations, presenting the results and analysing them in detail. Lastly, Section 4 concludes the paper, summarizing the key findings from the study.

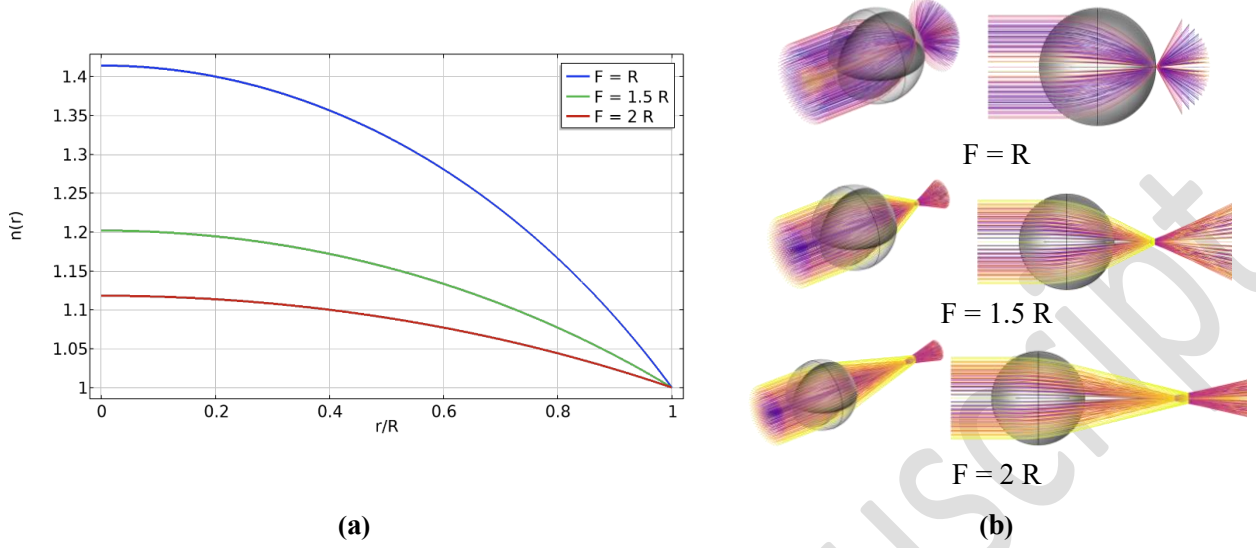
## 2. Design and Simulation Principles

Among acoustic lenses, gradient refractive index lenses (GRIN) have the property of spatially variable refractive index within the metamaterial and therefore easily manipulating the incident waves. Luneburg lens (originally in optics) is a symmetric gradient-index lens that can focus acoustic waves from any direction to an arbitrary point and thus is chosen as a suitable lens for this study[21]. The governing equation of the modified acoustic Luneburg lens[10] presents the refractive index  $n(r)$  at any point within the Luneburg lens as a function of the radial distance  $r$  from the center of the lens[10]:

$$n(r) = \frac{\sqrt{F^2 + R^2 - r^2}}{F} \quad (1),$$

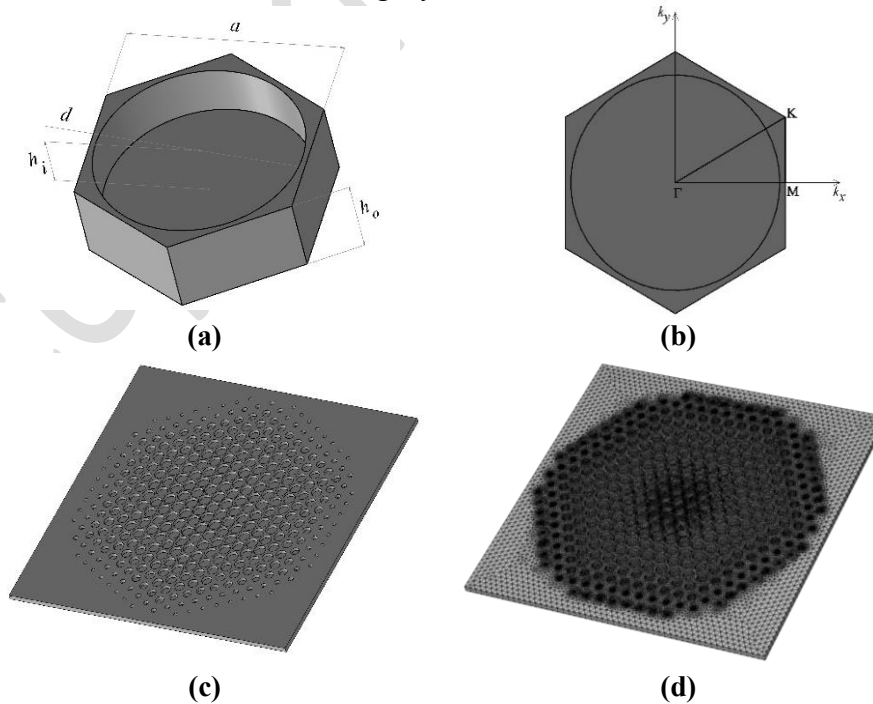
where  $R$  is the radius of the lens and  $F$  is the location of the focal point. The traditional profile of Luneburg lens can be achieved by using  $F=R$  in Eq. (1). Based on this governing equation which is plotted in Figure 1.a for three cases of  $F = R, 1.5R, 2R$ , the refractive index is highest at the center of the lens and decreases radially towards the edge. Low refractive index in outer boundaries results in impedance matching with background medium. Also, the smooth transition of refractive index allows sound waves to travel along curved paths within the lens. Therefore, the acoustic waves can enter and travel within the lens without any reflection. Figure 1.b is obtained by ray trajectory simulations using COMSOL's geometrical optics interface. A collimated array of rays is released into the sphere in which its only needed material property is defined with the modified Luneburg refractive index profile of Eq. (1). This figure shows how waves behave in each of the three mentioned focal distance cases. It can be seen that changing the refractive index profile from  $F=R$  to  $F=1.5R$  or  $F=2R$  results in the formation of an extended focal region rather than a sharp focal point. This indicates that the modified Luneburg lens inherently produces a broader convergence zone as the target focal distance increases. Theoretically, this behaviour is the direct result of flattening the refractive index profile when moving the focal point farther from the lens surface as indicated in Figure 1.a. A lower refractive index variation across the lens causes less

bending of the incident waves, leading to a longer convergence path. However, extended focal region is accepted in current study to reach the goal of acoustic metamaterial lenses for elastic waves with predetermined focal points, and further investigations similar to the one presented by Zhao et al. [10] for acoustic waves are required in future studies of this context to optimize the lens and reduce this aberration.



**Figure 1.** a) Refractive index profile of modified Luneburg lenses. b) Ray trajectories of Luneburg lens with various focal points.

To create a practical Luneburg lens, a unit cell able to create spatially varying refractive index should be proposed. For this reason, many concepts have been introduced by means of material or size variations. Examples include periodically arranged cross shape cells[10], variable thickness plates[15], and perforated plates with different hole sizes[16]. As demonstrated by Figure 2.a, hexagonal unit cells with blind holes of various diameters in the middle are proposed in this study. This choice results in a hexagonal final shape of the lens (Figure 2.b) which is a close shape to a circular Luneburg lens. The use of a blind hole design, rather than a basic perforated plate, enhances the structural integrity of the lens.



**Figure 2.** a) Hexagonal unit cell. b) Irreducible Brillouin zone. c) Luneburg lens designed based on refractive index profile with  $F=R$ . d) Meshed geometry of the lens.

The refractive indices of unit cells should be calculated to then properly arrange the cells next to each other and form the complete lens. This is done by deriving dispersion curves, which provide insight into how

waves propagate through the periodic structure at different frequencies. Calculating the dispersion curves involves solving the eigenvalue problem derived from the Floquet-Bloch theorem applied to the wave equation in solids. The wave equation for elastic wave propagation in a solid is given by[22]:

$$\nabla \cdot (C : \nabla u) = \rho \frac{\partial^2 u}{\partial t^2} \quad (2),$$

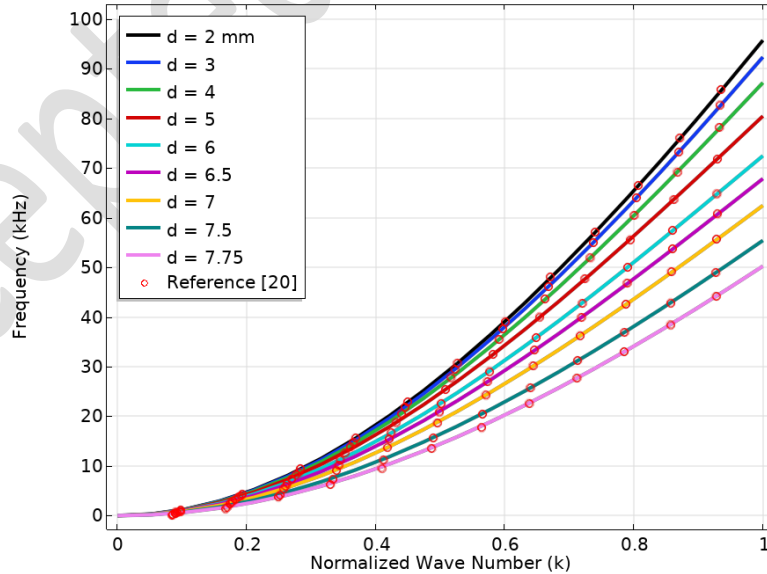
where  $C$  is the stiffness tensor,  $u$  is the displacement vector, and  $\rho$  is the density of the material. According to the Floquet-Bloch theorem, the solution to the wave equation in a periodic structure can be expressed as[23]:

$$u(r) = u_p(r) e^{ik \cdot r} \quad (3),$$

where  $u_p(r)$  is a periodic function with the same periodicity as the crystal structure and satisfies  $u_p(r+a) = u_p(r)$  with  $a$  as lattice constant, and  $e^{ik \cdot r}$  is a phase factor with  $k$  representing the wavevector. Substituting this form into the wave equation with periodic coefficients results in an eigenvalue problem, which is solved to obtain the dispersion relation  $\omega = \omega(k)$ . The computation of the dispersion relation is performed within the first Brillouin zone, which is the fundamental region in reciprocal ( $k_x$ - $k_y$ ) space for periodic structures. Due to the symmetry of the crystal, it is sufficient to compute the dispersion relation within the irreducible part of the Brillouin zone, rather than the entire lattice, which significantly reduces computational effort. This zone has all the unique wave propagation behavior of the structure, making it possible to predict wave behavior across the entire periodic medium.

The dispersion curves are calculated (shown and verified with reference in Figure 3) using finite element method (FEM) simulations in COMSOL Multiphysics. The eigenvalue problem is solved numerically over the irreducible Brillouin zone (IBZ). The IBZ is shown in Figure 2.b. By parametric sweeping the wave number of  $k_x$  along  $\Gamma M$  from  $0$  to  $\pi/a$ , eigenfrequencies are computed at each wave number value. This eigenfrequency analysis is performed for each unit cell under Floquet periodic boundary conditions, where a phase shift corresponding to wave vector of  $k_x$  is imposed between opposite sides of the unit cell. For each wave number of  $k_x$ , this process yields the natural frequencies that the unit cell can resonate with. In other words, the dispersion curve contains the allowable frequency-wave vector combinations inside the unit cell. Changing the blind hole diameters results in variations of these combinations and therefore each unit cell with a set blind hole diameter has a specific dispersion curve.

The hexagonal unit cell is considered with constant dimensions of short diagonal distance  $a = 8 \text{ mm}$ , plate thickness (outer side)  $h_o = 3.175 \text{ mm}$ , and hole depth (inner side)  $h_i = 2.175 \text{ mm}$ . However, diameter of the blind hole ( $d$ ) is considered as design variable and is changed to achieve a desirable refractive index.

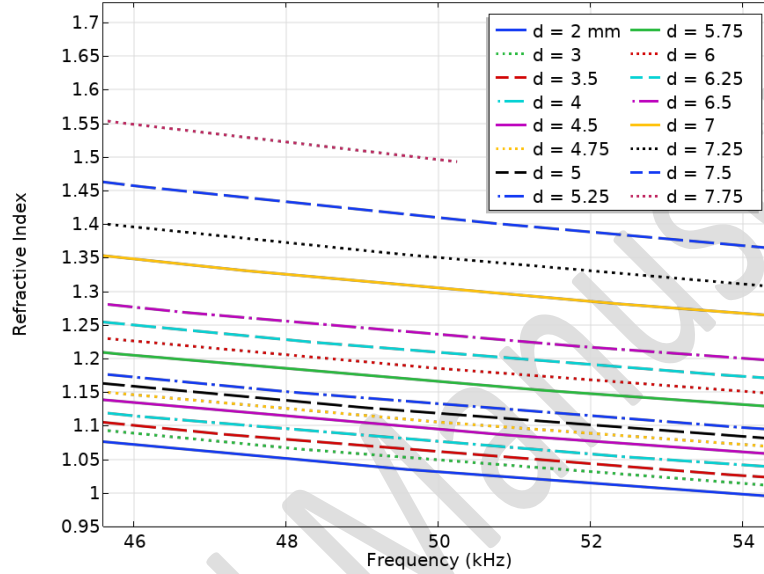


**Figure 3.** Dispersion diagram of unit cells of different diameter blind hole.

By examining the relationship between the wave vector and frequency (dispersion relation), the material's dispersion properties like wave propagation speed can be determined. The refractive index can then be calculated using Eq. (4) as follows[20]:

$$n = \frac{c}{c_{\Gamma M}} \quad (4),$$

in which  $c_{\Gamma M}$  is the phase velocity in the  $\Gamma M$  direction of the unit cell and  $c$  is the reference phase velocity of the first antisymmetric wave mode in a homogenous aluminium plate of the same thickness as the lens. The first antisymmetric mode represents the fundamental mode of wave propagation at the design frequency. The refractive indices of the unit cells are calculated using Eq. (4), with phase velocities extracted from the dispersion relations as  $c_{\Gamma M} = \omega(k_x)/k_x$ . Specifically, the phase velocity of the wave propagating through each unit cell is determined by the slope of its dispersion curve. Since  $\omega$  varies nonlinearly with  $k_x$  due to structure's internal resonances, the phase velocity changes with frequency. As a result, the refractive index becomes frequency dependent. Physically, the internal resonances are directly related to the modified effective mass and stiffness resulting from the introduction of blind holes in the structure of unit cell.



**Figure 4.** Refractive index per frequency for unit cells of different diameter blind hole.

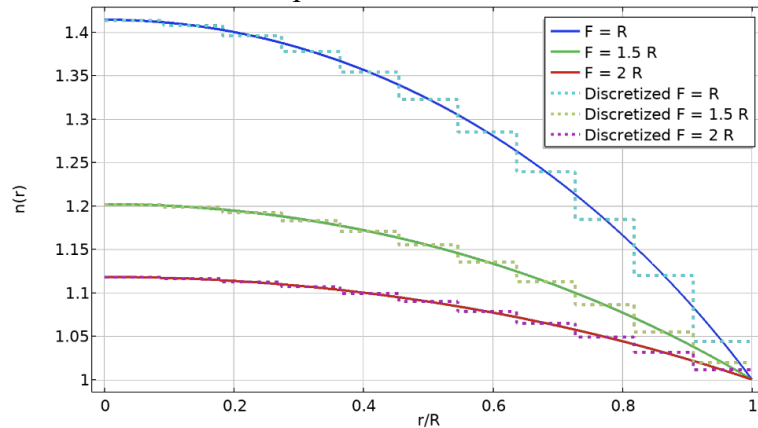
By analysing dispersion curves, the phase velocity values are obtained and used to compute the refractive indices and are presented in Figure 4. As illustrated, the refractive indices are strongly frequency dependent. The lowest refractive indices belong to unit cell of  $d = 2 \text{ mm}$  and by increasing the diameters of blind holes, the refractive index values also increase with the unit cell of  $d = 7.75 \text{ mm}$  having the highest values. Figure 4 directly results from the dispersion relation analysis. Dispersion diagram of Figure 3 depicts that longitudinal shape modes along the  $\Gamma M$  path exist only up to around  $50 \text{ kHz}$  in the  $7.75 \text{ mm}$  unit cell; beyond this frequency, other modes may emerge or vanish. This explains why the refractive index line in Figure 4 vanishes past  $50 \text{ kHz}$ —above that point, the longitudinal  $\Gamma M$  modes no longer exist.

**Table 1.** Refractive index of unit cells of different hole diameters at design frequency of  $50 \text{ kHz}$ .

| Diameter (mm) | Refractive Index |
|---------------|------------------|
| 2             | 1.0314           |
| 3             | 1.0492           |
| 4             | 1.0768           |
| 5             | 1.1185           |
| 6             | 1.1853           |
| 6.5           | 1.2360           |
| 7             | 1.3051           |
| 7.5           | 1.4096           |
| 7.75          | 1.4959           |

Table 1 indicates a broad range of refractive indices, from 1.03 to 1.4959, achievable at the design frequency ( $50 \text{ kHz}$ ) with the proposed unit cells. This wide range is particularly appropriate, because it allows for the precise design of modified Luneburg lens profiles through careful arrangements of unit cells next to each other in multiple concentric layers.

Based on the governing equation of the modified Luneburg lens, three hexagonal lenses are designed for three different focal points ( $F=R$ ,  $1.5R$ ,  $2R$ ). One of the lenses ( $F = R$ ) is shown in Figure 2.c. The three aforementioned modified Luneburg lens profiles are discretized to 11 layers as shown in Figure 5. The arrangement of the unit cells next to each other in concentric layers is based on refractive index of each unit cell and the required position of that refractive index in the profile.



**Figure 5.** Discretized Refractive index profile of Luneburg lens.

**Table 2.** Blind hole diameters (mm) in each layer for three designed lenses.

| Layers | $F = R$ | $F = 1.5R$ | $F = 2R$ |
|--------|---------|------------|----------|
| 1      | 7.5     | 6.25       | 5        |
| 2      | 7.5     | 6.25       | 5        |
| 3      | 7.5     | 6          | 5        |
| 4      | 7.25    | 6          | 4.75     |
| 5      | 7.25    | 5.75       | 4.75     |
| 6      | 7       | 7.5        | 4.5      |
| 7      | 7       | 5.25       | 4        |
| 8      | 6.5     | 5          | 3.5      |
| 9      | 6       | 4          | 3        |
| 10     | 5       | 3          | 2        |
| 11     | 3       | 2          | 2        |

The unit cells with largest blind holes are located around the center of the lens and small holes are located toward the edge of the lens. All three lenses have a diameter of  $176 \text{ mm}$  ( $R = 88 \text{ mm}$ ) consisting of 379 unit cells in 11 layers. Table 2 includes the corresponding diameters of blind holes in each layer of the three lenses.

The performance of the lens in focusing elastic waves is evaluated by means of finite element simulations in COMSOL Multiphysics software. The solid mechanics module is employed to perform a structural analysis in the frequency domain, allowing for derivation of the displacement, velocity, and stress fields. Frequency domain study (at design frequency of  $50 \text{ kHz}$ ) is specifically used to compute the response of the wave equation in solids to a harmonic wave with respect to particular load distributions. This provides the displacement field ( $u$ ). The velocity field can then be derived directly by applying the relation  $v=i\omega u$ , where  $v$  is the velocity field,  $\omega$  is the angular frequency.

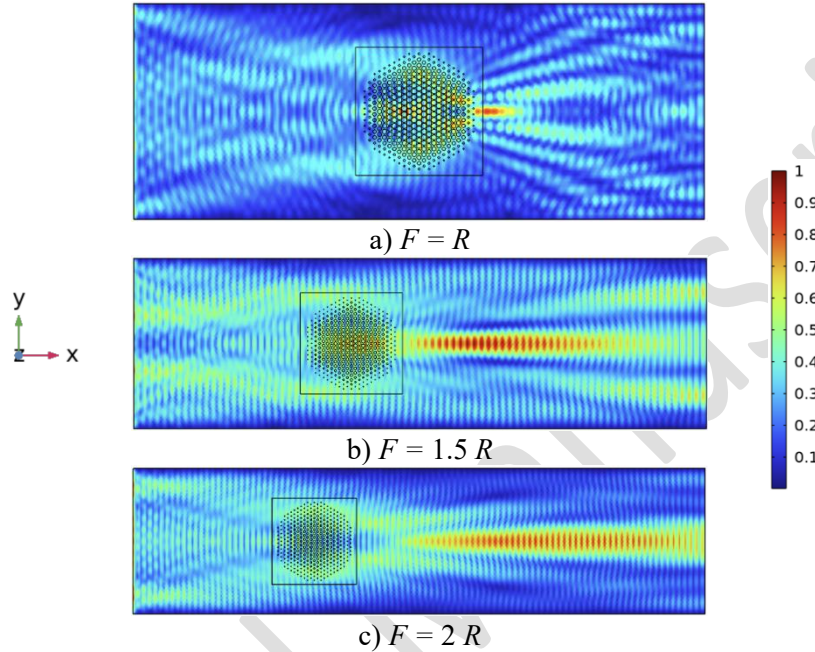
Simulation domain is chosen as a plate of  $800 \times 300 \text{ mm}^2$  area for  $F = R$ , a plate of  $1000 \times 300 \text{ mm}^2$  area for  $F = 1.5R$  and a plate of  $1200 \times 300 \text{ mm}^2$  area for  $F = 2R$  with a thickness of  $3 \text{ mm}$  in all cases. Aluminium with constants of elasticity  $E = 70 \text{ GPa}$ , density  $\rho = 2700 \text{ kg/m}^3$  and Poisson's ratio of  $\nu = 0.33$  is considered as the material for this study. Boundary load of  $F_z = 1 \text{ N}$  is applied to the left side of the plate and other sides are considered as low reflection boundary. Mesh size is set to be at least 7 mesh elements per wavelength in the design frequency of  $50 \text{ kHz}$ . An image of the meshed geometry of the lens is provided in Figure 2.d.

### 3. Results

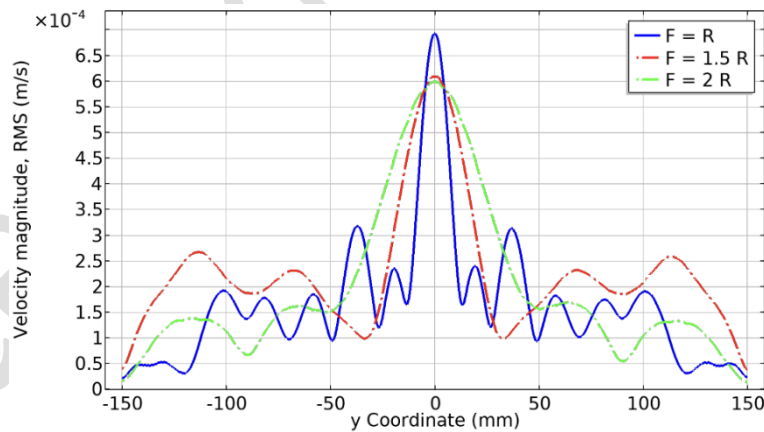
The normalized root-mean-square (RMS) velocity fields of Figure 6 are obtained as a result of simulations. As shown by this figure, elastic wave focusing is obtained by the proposed lenses of this study at three different focal points. In all three cases, the waves are curved toward the focal point of the lens, confirming the

lens's gradient refractive index profiles. It can be observed that wave concentration is stronger at focal points closer to the lens. Due to inherent aberrations of focal points in modified Luneburg lenses, which was pointed out in section 2, as the focal point moves from  $R$  to  $2R$ , the focal length increases accordingly. This brings out the possibility of using this type of lens for creating acoustic/elastic wave jets.

Figure 7 illustrates the RMS value of velocity over  $y$  coordinate. The  $xyz$  coordinate is placed in the middle of the upper surface of the lens with  $y$  being parallel to width of the domain ( $y$  is ranging from  $-150$  to  $150$  mm to cover  $300$  mm). As shown by this graph, focusing elastic wave is successfully achieved by the proposed lens at the vicinity of the intended focal points. The distribution of waves is being scattered along the  $y$  axis with an increase in focal point, and lens with  $F=R$  shows the highest level of focusing elastic waves based on the maximum velocity that is reached.



**Figure 6.** Normalized RMS velocity fields in simulation domain<sup>1</sup>.

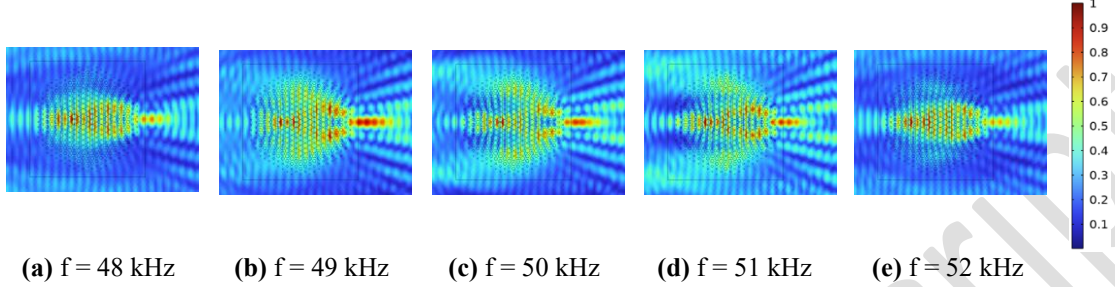


**Figure 7.** RMS velocity over  $y$  coordinate.

The three lenses were designed for the working frequency of  $50$  kHz. To further study the performance of these lenses, a frequency study was conducted from  $46$  kHz to  $52$  kHz and results are presented in Figure 8 and Table 3. This frequency analysis focuses on three key parameters of focusing: maximum amplification of RMS velocity with respect to reference velocity captured without the lens, full width at half maximum ( $FWHM$ ), and decay length estimated from the point of maximum velocity to  $\frac{1}{2}$  of this value. The amplification data indicates that significant focusing behavior occurs over the tested range in all three lenses. For the lens with focal point of  $F=R$ , the acceptable amplification (at least 1.5 times the reference velocity) happens in the range of  $49$  to  $51$  kHz. These values for  $1.5R$  and  $2R$  lenses are  $46$  to  $51$  kHz and  $47$  to  $51$  kHz respectively. The lens designed to

<sup>1</sup> Scales in Figure 6 are set to a) 1:8 b) 1:10 c) 1:12.

focus at  $F=R$  achieves the highest amplification of 5.96 at 50 kHz, with a subwavelength focusing of  $FWHM=0.73\lambda$  and a decay length of  $2.14\lambda$ . This lens has the narrowest focal zone with the lowest value for  $FWHM = 0.71\lambda$  at 51 kHz. Other two designed lenses with focal points of  $1.5R$  and  $2R$  exhibit high amplifications of 4.81 and 4.77 at the design frequency respectively. Figure 8 illustrates the normalized RMS velocity fields for the lens designed with a focal point of  $F=R$  across the frequency range of 48 kHz to 52 kHz. These images were selected as representative examples because this lens exhibited the highest amplification and sharpest focal zone among the three designs.



**Figure 8** Normalized RMS velocity fields of  $F=R$  Lens.

**Table 3.** Frequency-dependent analysis of the three acoustic metamaterial Luneburg lenses.

| Maximum Amplification of RMS Velocity |        |                 |      |                 |      |                 |      |      |
|---------------------------------------|--------|-----------------|------|-----------------|------|-----------------|------|------|
| Frequency (kHz)                       |        | 46              | 47   | 48              | 49   | 50              | 51   | 52   |
| Focal Point                           | F=R    | -               | -    | 1.02            | 3.32 | 5.96            | 2.86 | 1.4  |
|                                       | F=1.5R | 2.67            | 2.27 | 3.88            | 4.28 | 4.81            | 2.34 | 1.2  |
|                                       | F=2R   | 1.3             | 3.06 | 4.13            | 4.51 | 4.77            | 1.6  | 1.29 |
| FWHM                                  |        |                 |      |                 |      |                 |      |      |
| Frequency (kHz)                       |        | 49              |      | 50              |      | 51              |      |      |
| Focal Point                           | F=R    | 0.78 $\lambda$  |      | 0.73 $\lambda$  |      | 0.71 $\lambda$  |      |      |
|                                       | F=1.5R | 1.69 $\lambda$  |      | 1.64 $\lambda$  |      | 1.64 $\lambda$  |      |      |
|                                       | F=2R   | 3.74 $\lambda$  |      | 2.54 $\lambda$  |      | 2.57 $\lambda$  |      |      |
| Decay Length                          |        |                 |      |                 |      |                 |      |      |
| Frequency (kHz)                       |        | 49              |      | 50              |      | 51              |      |      |
| Focal Point                           | F=R    | 2.2 $\lambda$   |      | 2.14 $\lambda$  |      | 1.74 $\lambda$  |      |      |
|                                       | F=1.5R | 10.76 $\lambda$ |      | 10.32 $\lambda$ |      | 9.87 $\lambda$  |      |      |
|                                       | F=2R   | 15.63 $\lambda$ |      | 15.02 $\lambda$ |      | 14.42 $\lambda$ |      |      |

Although decent amplification is observed in the frequency range under study, correct Luneburg lens's functionality defined as sharp focusing at the pre-designed focal points, is only acceptable between 49 kHz and 51 kHz. In this narrower band, all three lenses focus waves at their pre-intended focal points. For instance, as the frequency changes to 46 kHz, the lens intended for  $F=1.5R$  still has a high amplification of 2.67 at that position, but the peak amplification no longer happens at the expected location, instead the peak happens to be closer to the lens. This deviation of focal point is due to the dispersive behavior of the metamaterial's unit cells. As shown in Figure 4, unit cells' refractive indices are frequency-dependent. As a result, the refractive index profiles no longer match the intended focal point's Luneburg profile. For example, a unit cell with a blind hole diameter of 3 mm that exhibits an index of 1.05 at 50 kHz, shifts this value to 1.09 at 46 kHz, resulting in a reconfigured refractive index profile of the lens. The waves will no longer concentrate at the intended focal point but instead focus at a new and unwanted focal point. While this still results in significant focusing and amplification, the lens no longer functions as the design intended. On the other hand, if achieving a long focal length is desirable in applications of the lenses instead of the exact position of focal point, the lenses can be assumed to work in even a broader frequency range. These lenses can achieve high decay length values of up to  $15.63\lambda$  in the case of  $F=2R$ , which is excellent for vibration-based energy harvesting techniques and acoustic jet applications. Therefore, while the designed lenses exhibit excellent broadband behavior and can still produce high amplification across a wider frequency range, their intended functionality is constrained to a narrower 2 kHz band around the design frequency of 50 kHz.

Comparing the results of this study to previous works is difficult due to different objectives of reported metamaterial lenses for elastic waves, but several studies can still be useful references for this purpose. Table 4 provides a comparison of available key performance parameters such as operating frequency, full width at half maximum (*FWHM*), amplification, and lens design method. Previous studies focused primarily on lens design methods with a fixed location of the focal point and did not investigate methods for passively positioning the focus at different locations like the one presented in this study. In contrast, the design approach presented in this study enables arbitrary placement of the focal point through structural modification. The lens designed for  $F=R$  achieved the highest amplification among the compared works, along with subwavelength focusing. The results presented here show higher amplification values even for  $F=1.5R$  and  $2R$  lenses. Although the *FWHM* values for these lenses are not as sharply focused and subwavelength as others, they still represent localized wave concentrations at their focal points. This highlights the effectiveness of the proposed design method.

**Table 4.** Comparison of the reported results from acoustic metamaterial lenses for elastic waves focusing.

| Reference                                | Frequency (kHz)                            | FWHM          | Amplification | Design Method   |
|--|--|---------------|---------------|---|
| Yang et al. [16]                         | 16-19<br>(Results for 16 kHz)              | -             | 3             | Graded negative refractive index plate lens.  |
| Quadrelli et al. [17]                    | 0.321                                      | $0.59\lambda$ | -             | Double-negative piezoelectric metamaterial.   |
| Zhao et al. [18]                         | 50, 100, 150, 200<br>(Results for 100 kHz) | $0.5\lambda$  | 4             | Structural Generalized Luneburg lens (GLL), localization of energy between 2 focal spots.   |
| Tol et al. [19]                          | 50   | -             | 3.7           | Gradient-Index Phononic Crystal Lens (GRIN-PCL) with hyperbolic secant gradient refractive index profile.                           |
| Current study for the lens with $F=R$    | 49-51<br>(Results for 50 kHz)              | $0.73\lambda$ | 5.96          | Adjustable focal point design method with gradient-index acoustic metamaterial based on modified Luneburg refractive index profile. |
| Current study for the lens with $F=1.5R$ | 49-51<br>(Results for 50 kHz)              | $1.64\lambda$ | 4.81          |   |
| Current study for the lens with $F=2R$   | 49-51<br>(Results for 50 kHz)              | $2.54\lambda$ | 4.77          |   |

## 4. Conclusion

This paper demonstrates the novel design method of acoustic lenses with predetermined focal points of elastic waves based on the modified Luneburg refractive index profile. This is achieved by the design and simulation of three lenses for focusing elastic waves at three different focal points. The hexagonal unit cells presented in this study have design parameters which enables them to achieve a wide range of refractive indices, ranging from 1.0314 to 1.4959 at the design frequency of 50 kHz. Dispersion curves are calculated via finite element method simulations, and as a result the refractive index is derived for each unit cell at the design frequency. The lenses are formed by arranging hexagonal unit cells with blind holes of varying hole diameters based on modified Luneburg lens refractive index profiles. The results from numerical wave simulations confirmed the accurate focusing of waves at the intended focal points of  $F = R$ ,  $1.5R$ , and  $2R$ , validating the effectiveness of the design. The lens designed for  $F = R$  achieved the highest level of focusing based on maximum velocity values with the highest amplification of 5.96 times the reference velocity. As the focal point shifted from  $R$  to  $2R$ , the wave distribution became scattered along the focal axis. Furthermore, a frequency-dependent study from 46 kHz to 52 kHz revealed that precise Luneburg lens focusing is achieved within 49–51 kHz band. Outside this band, significant amplification is observed while focal points shift due to refractive index profile changes. The  $F = R$  lens produces a subwavelength focal zone, achieving a minimum *FWHM* of  $0.71\lambda$  at 51 kHz, whereas the  $F = 1.5R$  and  $F = 2R$  designs yield focal widths of  $1.6\lambda$  and  $2.6\lambda$ . High values of decay lengths are achieved with these lenses. For instance, a decay length of up to  $15.63\lambda$  with the  $F = 2R$  lens, emphasises its potential for applications requiring a long focal region. The adaptability and structural integrity of the proposed design make this approach a promising path for advanced wave manipulation. The results of this study offer significant potential for applications in areas such as industrial non-destructive testing, imaging, and energy harvesting.

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